Mathematics 411: Advanced Calculus I
Model Solutions to the Final Examination

Problem 1: Suppose that $X, Y \subset \mathbf{R}$ are nonempty and satisfy $(\forall x \in X)(\forall y \in Y) x < y$.
(a) Prove that real numbers $\sup X$ and $\inf Y$ exist.
(b) Prove that $\sup X \le \inf Y$.
(c) Exhibit example X, Y for which $\sup X = \inf Y$.

Solution 1: (a) Since Y is nonempty, we may pick some $y_0 \in Y$ to have $(\forall x \in X) x < y_0$. Thus X is bounded above, as well as nonempty, so $\sup X \in \mathbf{R}$ exists. Similarly, we may pick some $x_0 \in X$, which is nonempty, to have $(\forall y \in Y) y < x_0$. Thus Y is bounded below, as well as nonempty, so $\inf Y \in \mathbf{R}$ exists.

(b) Let $\epsilon > 0$ be given. By the approximation theorem, we may find $y_0 \in Y$ satisfying $\inf Y > y_0 - \epsilon$. Since $(\forall x \in X) x < y_0$, we conclude that $(\forall x \in X) x < \inf Y + \epsilon$, so $\sup X \le \inf Y + \epsilon$. But ϵ was arbitrary, so $\sup X \le \inf Y$.

(c) Let $X = \mathbf{R}^-$ and $Y = \mathbf{R}^+$. Then $\sup X = \inf Y = 0$. ∎

Problem 2: Let $\{x_n : n = 1, 2, \dots\} \subset S$ be a sequence in a metric space (S, d), and suppose that $x_n \to x \in S$ as $n \to \infty$. Prove that $\{x_n\}$ is a Cauchy sequence.

Solution 2: Given $\epsilon > 0$, choose N such that $n \ge N \implies d(x_n, x) < \epsilon/2$. This N exists by definition, since $x_n \to x \in S$ as $n \to \infty$. Then by the triangle inequality, $d(x_n, x_m) \le d(x_n, x) + d(x_m, x)$, so

$$n, m \ge N \implies d(x_n, x_m) \le d(x_n, x) + d(x_m, x) < \epsilon.$$

Since such an N exists for every $\epsilon > 0$, $\{x_n\}$ is a Cauchy sequence. ∎

Problem 3: Prove that the function $e^{\sin(x^2)}$ has bounded variation on any bounded interval $[a, b]$.

Solution 3: The function $f(x) = e^{\sin(x^2)}$ has a continuous derivative $f'(x) = 2x\cos(x^2)e^{\sin(x^2)}$ that is bounded by $|f'(x)| \le 2e \max\{|a|, |b|\}$ on the bounded interval $[a, b]$, and thus has bounded variation on $[a, b]$. ∎

Problem 4: Suppose that $f = f(x)$ and $g = g(x)$ are both Riemann integrable on $[a, b]$ and that $g(x) > 1$ and $|f(x)| \le 1$ for every $x \in [a, b]$.
(a) Prove that f^2/g is Riemann integrable on $[a, b]$.
(b) Prove that $\int_a^b (f^2/g)\, dx \le b - a$.

Solution 4: (a) First note that since f is Riemann integrable on $[a, b]$, then so is f^2. Next, since $g(x) > 1$ is bounded away from 0 on $[a, b]$, by Theorem 7.49 f^2/g is Riemann integrable on $[a, b]$.

(b) Since $\alpha(x) = x$ is an increasing integrator, we may use comparison Theorem 7.20. Since $f(x)^2/g(x) \le 1^2/1 = 1$ on $[a, b]$, we conclude that $\int_a^b (f(x)^2/g(x))\,dx \le \int_a^b 1\,dx = b - a$. ■

Problem 5: Consider the following function defined on $[0, 1]$:

$$f(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \in (0,1) \text{ is irrational;} \\ 1/n^2, & \text{if } x \in (0,1] \text{ is rational, with } x = m/n \text{ in lowest terms.} \end{cases}$$

(a) Compute the oscillation $\omega_f(x)$ of f at each $x \in [0, 1]$.
(b) Determine, with proof, whether f is Riemann integrable on $[0, 1]$.

Solution 5: (a) We suppose that $x \in [0, 1]$ and compute

$$\omega_f(x) = \lim_{h \to 0+} \sup\{|f(s) - f(t)| : s, t \in B(x; h)\} = \begin{cases} 0, & \text{if } x = 0 \text{ or } x \text{ is irrational,} \\ 1/n^2, & \text{if } x = m/n \text{ in lowest terms.} \end{cases}$$

(b) By Exercise 4.24 of the text, f is continuous except at the rational points in $(0, 1]$, which are countable and thus form a set of measure zero. We conclude by Lebesgue's criterion (Theorem 7.48, p.171) that f is Riemann integrable on $[0, 1]$. ■

Problem 6: Fix $[a, b] \subset \mathbf{R}$ and suppose that $\mathbf{f}(t) = (t, \cos \pi t, \sin \pi t) \in \mathbf{R}^3$ for $a \le t \le b$. Prove that $\mathbf{f}$ is a rectifiable path in $\mathbf{R}^3$, and compute its length $\Lambda_{\mathbf{f}}(a, b)$ over $[a, b]$.

Solution 6: Since $\mathbf{f}$ has a continuous derivative $\mathbf{f}'(t) = (1, -\pi \sin t, \pi \cos t)$, it is a rectifiable path over any closed and bounded interval $[a, b]$. To compute its length, we may use the formula

$$\Lambda_{\mathbf{f}}(a, b) = \int_a^b \|\mathbf{f}'(t)\|\,dt = \int_a^b \sqrt{1 + \pi^2}\,dt = (b - a)\sqrt{1 + \pi^2},$$

since $\|\mathbf{f}'(t)\| = \sqrt{1^2 + \pi^2 \sin^2 t + \pi^2 \cos^2 t} = \sqrt{1 + \pi^2}$. ■

Problem 7: Let $C \subset [0, 1] \subset \mathbf{R}$ be the *Cantor set*, namely those numbers c expressible as $c = \sum_{k=1}^{\infty} a_k/3^k$, with $a_k \in \{0, 2\}$ for every k.
(a) Prove that C is uncountable.
(b) Prove that C is a set of measure zero.

Solution 7: (a) Define the function $f : C \to [0, 1]$ by $f(c) = \sum_{k=1}^{\infty} [a_k/2]/2^k$, which converts the base-3 expansion of c into a base-2 expansion. Any base-2 expansion $\{b_k\}$ is obtained from some c with $a_k = 2b_k$, so f maps C onto $[0, 1]$, so there is a 1-1 correspondence between the uncountable set $[0, 1]$ and some subset of C. Hence C has an uncountable subset, and must itself be uncountable.

(b) Let $\epsilon > 0$ be given and choose N such that $(2/3)^N < \epsilon$. Let $C_N \subset [0, 1]$ be the set of numbers c expressible as $c = \sum_{k=1}^{\infty} a_k/3^k$, with $a_k \in \{0, 2\}$ for every $1 \le k \le N$, and $a_k \in \{0, 1, 2\}$ for $k > N$. Then $C \subset C_N$ for every N. We claim that C_N is the disjoint union of closed intervals of total length $(2/3)^N < \epsilon$. These intervals are of the form $[x, x + 3^{-N}]$, where $x = .a_1 \cdots a_{N-1}0$ (base 3) or $x = .a_1 \cdots a_{N-1}2$ (base 3). Since each a_k can be either 0 or 2, there are $2 \times 2^{N-1} = 2^N$ such intervals, and each has length 3^{-N}, giving the result.