Mathematics 412: Advanced Calculus II
Problem Set 4 — due Thursday, February 28, 2002

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Verify that the trigonometric system $\{\frac{1}{\sqrt{\pi}}\sin nx : n = 1, 2, \ldots\} \cup \{\frac{1}{\sqrt{\pi}}\cos nx : n = 1, 2, \ldots\} \cup \{\frac{1}{\sqrt{2\pi}}\}$ is orthonormal on $[0, 2\pi]$.

Problem 2: Let $\{\phi_0, \phi_1, \ldots\} \subset X$ be an orthonormal system of functions, where X is an inner product space in which $\|f\| = 0 \iff f = 0$. Prove that the following three statements are equivalent:

(a) If $\langle f, \phi_n\rangle = \langle g, \phi_n\rangle$ for all $n = 0, 1, 2, \ldots$, then $f = g$.

(b) If $\langle f, \phi_n\rangle = 0$ for all n, then $f = 0$.

(c) If T is an orthonormal system such that $\{\phi_0, \phi_1, \ldots\} \subset T$, then $T = \{\phi_0, \phi_1, \ldots\}$.

For Problems 3 and 4, define $C([0, 1])$ to be the space of complex-valued continuous functions on the compact interval $[0, 1]$, with $\langle f, g\rangle \stackrel{\text{def}}{=} \int_0^1 f(t)\bar{g}(t)\,dt$ for $f, g \in C([0, 1])$, and $\|f\| \stackrel{\text{def}}{=} \sqrt{\langle f, f\rangle}$.

Problem 3: For $f \in C([0, 1])$, prove that $\|f\| = 0 \iff f = 0$.

Problem 4: Prove that the set $\{e^{2\pi int} : n \in \mathbf{Z}\} \subset C([0, 1])$ is an orthonormal system satisfying all three conditions of Problem 2.

Problem 5: Suppose that $f \in L([-\pi, \pi])$ and that f is 2π-periodic on $\mathbf{R}$. Show that:

(a) If $f(-x) = f(x)$ for $x \in [0, \pi]$, then $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where $a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$.

(b) If $f(-x) = -f(x)$ for $x \in [0, \pi]$, then $f \sim \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt$.

Remark. This means that we need to use just sines or just cosines for the Fourier series of a function defined on $L([0, \pi])$.

Problem 6: Show that $x = \pi - 2\displaystyle\sum_{n=1}^{\infty}\frac{\sin nx}{n}$, if $0 < x < 2\pi$. (Hint: don't use Solution 5.)

Problem 7: Show that $\dfrac{x^2}{2} = \pi x + 2\displaystyle\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 2\sum_{n=1}^{\infty}\frac{1}{n^2}$, if $0 \le x \le 2\pi$. Conclude that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. (Hint: integrate the formula from Problem 6.)

For Problems 8 and 9, define a 2π-periodic function f as follows:

$$f(t) = \begin{cases} 1, & \text{if } 0 < t < \pi; \\ -1, & \text{if } -\pi < t < 0; \\ 0, & \text{if } t = -\pi,\ t = 0, \text{ or } t = \pi. \end{cases}$$

Problem 8: Show that $f(x) = \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$ for every $x \in \mathbf{R}$.

Problem 9: Let $s_n(x)$ be the partial sum of the first n terms of the Fourier series of the function f defined above. Show that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \left[\max_{|x|<\epsilon} s_n(x) - \min_{|x|<\epsilon} s_n(x) \right] = \frac{4}{\pi} \int_0^{\pi} \frac{\sin t}{t}\, dt.$$

(Hint: see problem 11.19 on pp.338–339 of the text.)

Problem 10: Prove that if $f \in L([0, 2\pi])$ and $f'(x_0)$ exists at some point $x_0 \in (0, 2\pi)$, then the Fourier series generated by f converges at x_0.