Mathematics 412: Advanced Calculus II
Worked Solutions to Problem Set 7

PROF. M. V. WICKERHAUSER

Problem 1: Suppose that $f_i : \mathbf{R} \to \mathbf{R}$ is defined and bounded on the compact interval $[a_i, b_i] \subset \mathbf{R}$. If $f_i \in R([a_i, b_i])$ for $i = 1, \ldots, n$, prove that

$$\int_Q f_1(x_1) \cdots f_n(x_n)\, d(x_1, \ldots, x_n) = \left(\int_{a_1}^{b_1} f_1(x_1)\, dx_1 \right) \cdots \left(\int_{a_n}^{b_n} f_n(x_n)\, dx_n \right),$$

where $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbf{R}^n$.

Solution 1: We prove this by induction on n. It is evidently true when $n = 1$, for then both sides are the same.

Suppose the result holds for $n-1$; let $g : \mathbf{R}^{n-1} \to \mathbf{R}$ be defined by $g(x_1, \ldots, x_{n-1}) = f(x_1) \cdots f(x_{n-1})$. This g is defined and bounded on $Q_{n-1} \stackrel{\text{def}}{=} [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$. For any $f_n \in R([a_n, b_n])$, the function $gf_n = g(x_1, \ldots, x_{n-1}) f_n(x_n)$ belongs to $R(Q)$ by Lebesgue's criterion for Riemann integrability, and we have

$$\int_Q g(x_1, \ldots, x_{n-1}) f_n(x_n)\, d(x_1, \ldots, x_n) = \int_{Q_{n-1}} \left[\overline{\int_{a_n}^{b_n}} g(x_1, \ldots, x_{n-1}) f_n(x_n)\, dx_n \right] d(x_1, \ldots, x_{n-1}),$$

by Theorem 14.7, p.396. But $g(x_1, \ldots, x_{n-1})$ may be removed from the inner upper integral, since it has no x_n-dependence, and the remaining function f_n is Riemann integrable, so the upper integral may be replaced by a Riemann integral. The value of that integral may be factored out to give:

$$\int_Q g(x_1, \ldots, x_{n-1}) f_n(x_n)\, d(x_1, \ldots, x_n) = \left[\int_{a_n}^{b_n} f_n(x_n)\, dx_n \right] \int_{Q_{n-1}} g(x_1, \ldots, x_{n-1})\, d(x_1, \ldots, x_{n-1}),$$

and the result for n follows from the inductive hypothesis. ■

Problem 2: Let $Q = [0,1] \times [0,1]$ and $f(x, y) = x^2 + y^2$. Compute $\int_Q f(x, y)\, d(x, y)$.

Solution 2: Since f is defined, bounded, and continuous on Q, it is in $R(Q)$ by Lebesgue's criterion, so we may use interated integration and the Calculus to evaluate

$$\int_Q f = \int_0^1 \left(\int_0^1 [x^2 + y^2]\, dx \right) dy = \int_0^1 \left[\frac{1}{3}x^3 + xy^2 \right]_{x=0}^1 dy = \int_0^1 \left(\frac{1}{3} + y^2 \right) dy = \left[\frac{1}{3}y + \frac{1}{3}y^3 \right]_{y=0}^1 = \frac{2}{3}. \quad ■$$

Problem 3: Let $S = \{(x, y) \in \mathbf{R}^2 : |x| + |y| \leq 1, x \geq y \geq 0\}$ and put $f(x, y) = x^2 + y^2$. Compute $\int_S f(x, y)\, d(x, y)$.

Solution 3: The region is a right triangle with vertices $(0,0)$, $(1,0)$, and $(\frac{1}{2}, \frac{1}{2})$. Thus S is Jordan measurable, since its boundary is a finite union of line segments in $\mathbf{R}^2$. To use Theorem 14.13, we decompose

S into two subtriangles, which are likewise Jordan measurable, separated by the line $\{(x,y) : x = \frac{1}{2}\}$. Then, we use Theorem 14.14 to evaluate the integral of the continuous function f on each of the two parts:

$$\begin{aligned}
\int_S f &= \int_{x=0}^{\frac{1}{2}} \left(\int_{y=0}^{x} [x^2 + y^2]\, dy \right) dx + \int_{x=\frac{1}{2}}^{1} \left(\int_{y=0}^{1-x} [x^2 + y^2]\, dy \right) dx \\
&= \int_{x=0}^{\frac{1}{2}} \left[x^2 y + \frac{1}{3} y^3 \right]_{y=0}^{x} dx + \int_{x=\frac{1}{2}}^{1} \left[x^2 y + \frac{1}{3} y^3 \right]_{y=0}^{1-x} dx \\
&= \int_{x=0}^{\frac{1}{2}} \frac{4}{3} x^3\, dx + \int_{x=\frac{1}{2}}^{1} \left(\frac{1}{3} - x + 2x^2 - \frac{4}{3} x^3 \right) dx \\
&= \left[\frac{1}{3} x^4 \right]_{x=0}^{\frac{1}{2}} + \left[\frac{1}{3} x - \frac{1}{2} x^2 + \frac{2}{3} x^3 - \frac{1}{3} x^4 \right]_{x=\frac{1}{2}}^{1} = \frac{1}{48} + \frac{3}{48} = \frac{1}{12}. \quad \blacksquare
\end{aligned}$$

Problem 4: Let $Q = [0,1] \times [0,1] \subset \mathbf{R}^2$ and Define $f : Q \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} 0, & \text{if at least one of } x, y \text{ is irrational;} \\ 1/n, & \text{if } y \text{ is rational and } x = m/n, \end{cases}$$

where m, n are relatively prime nonnegative integers expressing the rational number x in lowest terms. Prove the following facts about Riemann integrals:

(a) $\int_0^1 f(x,y)\, dx = 0$ exists,

(b) $\int_0^1 \left[\int_0^1 f(x,y)\, dx \right] dy = 0$ exists,

(c) $\int_Q f(x,y)\, d(x,y) = 0$ exists, but

(d) $\int_0^1 f(x,y)\, dy$ does not exist for any rational $x \in [0,1]$.

Solution 4: For (a), note that for any fixed $y \in [0,1]$, the function $g(x) \stackrel{\text{def}}{=} f(x,y)$ is zero and continuous at each irrational $x \in [0,1]$. By Lebesgue's criterion, $g \in R([0,1])$, and $g = 0$ almost everywhere on $[0,1]$, so $\int_0^1 f(x,y)\, dx = 0$ exists. Then (b) follows immediately.

For (c), note that f is discontinuous only at points in Q with both coordinates rational. Elsewhere in Q, f is continuous and takes the value 0. Since the rational-coordinate points in Q form a countable set, they have measure zero, so $f \in R(Q)$ by Lebesgue's criterion. Since $f = 0$ almost everywhere in Q, we have $\int_Q f = 0$.

For (d), note that if $x = m/n$ in lowest terms, then the function $g(y) \stackrel{\text{def}}{=} f(x,y)$ takes the value 0 at irrational y, and $1/n \neq 0$ at rational y. But this function g is therefore discontinuous at each $y \in [0,1]$ and fails to be integrable by Lebesgue's criterion. $\blacksquare$

Problem 5: Suppose that $S \subset \mathbf{R}^n$ is a bounded set having finitely many accumulation points. Prove that $c(S) = 0$.

Solution 5: We will show that $(\forall \epsilon > 0)\bar{c}(S) < \epsilon$, which implies that $\bar{c}(S) = 0$ and thus $c(S) = 0$.

So, let $\epsilon > 0$ be given. Let $A = \{s_k : k = 1, \ldots, K\}$ be the accumulation points of $S \subset I$, where I is a bounded n-interval. We may construct a partition of I with subintervals of measure less than $\epsilon/(2^{n+1}K)$. Let Q_k^* be the union of all the subintervals containing $s_k \in A$. If s_k belongs to the interior of some subinterval, then Q_k^* consists of a single subinterval and has measure less that $\epsilon/(2^{n+1}K)$. If, however, s_k belongs to the boundary of some subinterval, then Q_k^* might have as many as 2^n adjacent components. Assuming this maximum number of components, we still have $\sum_{k=1}^K |Q_k^*| < 2^n K\epsilon/(2^{n+1}K) = \epsilon/2$.

By our construction, s_k belongs to $\operatorname{int} Q_k^*$ for each k, so we have constructed an open cover $A \subset \bigcup_{k=1}^K \operatorname{int} Q_k^* \stackrel{\text{def}}{=} Q^*$.

There can be at most finitely many, say $B = \{x_m \in S : m = 1, \ldots, M\}$, points of S outside Q^*, since an infinite set would have to accumulate somewhere outside Q^*, giving an accumulation point not in A. We now refine the original partition to insure that its subintervals have measure less than $\epsilon/(2^{n+1}M)$. For each point $x_m \in B$, let R_m^* be the union of all subintervals of the new partition containing x_m. There are at most 2^n of them, so $|R_m^*| < \epsilon/(2M)$ for each $m = 1, \ldots, M$. Let $R^* = \bigcup_{m=1}^M R_m^*$ be the union of these M bunches of subintervals covering B. The total measure of R^* is less than $M\epsilon/(2M) = \epsilon/2$.

But $\bar{S} \subset Q^* \cup R^*$, so $\bar{c}(S) \le |Q^*| + |R^*| < \epsilon$, and the proof is complete. ■

Problem 6: Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function defined on $[a, b] \subset \mathbf{R}$, and let $S = \{(x, y) \in \mathbf{R}^2 : x \in [a, b], y = f(x)\}$ be the graph of f. Prove that the two-dimensional Jordan content $c(S)$ is zero.

Solution 6: Since $f \in R([a, b])$, Riemann's integrability criterion applies. Given $\epsilon > 0$, choose a partition $P \in \mathcal{P}([a, b])$ such that $0 \le U(f, P) - L(f, P) < \epsilon$. But then the rectangles $\{I_k \times [m_k, M_k] : k = 1, \ldots, N\}$ have total area less than ϵ, where $I_k \stackrel{\text{def}}{=} [x_{k-1}, x_k]$ is one of the N subintervals of P, $m_k = \inf\{f(x) : x \in I_k\}$, and $M_k = \sup\{f(x) : x \in I_k\}$. These rectangles cover the graph S of f. Since ϵ was arbitrary, we have shown that $\bar{c}(S) = 0$, and since $0 \le c(S) \le \bar{c}(S)$, we conclude that $c(S) = 0$. ■

Problem 7: Let S be a bounded line segment in $\mathbf{R}^n$, $n \ge 2$. Prove that S has n-dimensional Jordan content zero.

Solution 7: First note that each pair of corner points $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{R}^n$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbf{R}^n$ determines an n-interval $[\mathbf{a}, \mathbf{b}] \stackrel{\text{def}}{=} |a_1, b_1| \times \cdots \times |a_n, b_n|$, where

$$|p, q| \stackrel{\text{def}}{=} \begin{cases} [p, q], & \text{if } p \le q, \\ [q, p], & \text{if } p > q. \end{cases}$$

Now, let $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ be the endpoints of S, so that $S = \{\mathbf{f}(t) \stackrel{\text{def}}{=} (1 - t)\mathbf{a} + t\mathbf{b} : 0 \le t \le 1\} \subset \mathbf{R}^n$.

We choose $N > 0$ and partition $[0, 1]$ into N equal subintervals $[\frac{k-1}{N}, \frac{k}{N}]$. This determines N equal n-subintervals $[\mathbf{f}\left(\frac{k-1}{N}\right), \mathbf{f}\left(\frac{k}{N}\right)]$, which each have volume less than $(\|\mathbf{b} - \mathbf{a}\|/N)^n$. These cover S, and since there are N of them, their total volume is less than $\|\mathbf{b} - \mathbf{a}\|^n/N^{n-1}$. Since $n > 1$, we can making this total volume as small as we like by choosing sufficiently large N. We conclude that $\bar{c}(S) = 0$, so $c(S) = 0$. ■

Problem 8: Define

$$f(x, y) = \begin{cases} e^{-xy} \sin x \sin y, & \text{if } x \ge 0 \text{ and } y \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

Prove that both iterated integrals exist, with

$$\int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x, y)\, dx \right] dy = \int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x, y)\, dy \right] dx,$$

but that the double integral of f over $\mathbf{R}^2$ does not exist. Why does this not contradict the Tonelli-Hobson test (Th.15.8, p.415)?

Solution 8: In fact this function is integrable over $\mathbf{R}^2$, by the Tonelli–Hobson test! We use a table of Laplace transforms such as Table 17.13, p.1178 of Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products*, ISBN 0-12-294755. Line 34 on p.1179 states:

$$\int_0^\infty e^{-xy}|\sin x|\,dx = \frac{\coth(\pi y/2)}{1+y^2},$$

so that

$$\int_{\mathbf{R}}\left[\int_{\mathbf{R}}|f(x,y)|\,dx\right]\,dy = \int_0^\infty \frac{|\sin y|\coth(\pi y/2)}{1+y^2}\,dy,$$

where $\coth\theta = (e^\theta + e^{-\theta})/(e^\theta - e^{-\theta}) \sim 1/\theta$ as $\theta \to 0$, while $\coth\theta \to 1$ as $y \to \infty$. Thus $\sin(y)\coth(\pi y/2) \sim 2\sin(y)/\pi y \to 2/\pi$ as $y \to 0$, and the integrand at right is continuous and dominated by $1/(1+y^2)$, which is Lebesgue integrable on $[0,+\infty)$. The Tonelli–Hobson test applies, and we conclude that $f \in L(\mathbf{R}^2)$. ■

Problem 9: Prove that $\int_{\mathbf{R}^2} e^{-x^2-y^2}\,d(x,y) = \pi$ by transforming to polar coordinates. Use this to prove that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Solution 9: We use the coordinate transformation $g : (r,\theta) \to (x,y) = (r\cos\theta, r\sin\theta)$, whose Jacobian determinant is r according to Example 1, p.418. Put $T = (0,+\infty)\times(0,2\pi)$ and $\mathbf{R}^1_+ \stackrel{\text{def}}{=} \{(x,0) \subset \mathbf{R}^2 : x \geq 0\}$. Then $g : T \to \mathbf{R}^2 \setminus \mathbf{R}^1_+$ is 1-1, onto, and continuously differentiable with nonzero Jacobian determinant. The omitted range set $\mathbf{R}^1_+$ has zero 2-measure, so by Theorem 15.11, p.421, we have

$$\int_{\mathbf{R}^2} e^{-x^2-y^2}\,d(x,y) = \int_{\mathbf{R}^2\setminus\mathbf{R}^1_+} e^{-x^2-y^2}\,d(x,y) = \int_T e^{-r^2}|r|.$$

Now $|r| = r$ for $r > 0$, and the function re^{-r^2} belongs to $L(T)$, so we may use Fubini's theorem to evaluate the integral as follows:

$$\int_T e^{-r^2}|r| = \int_0^{2\pi}\left(\int_0^\infty re^{-r^2}\,dr\right)\,d\theta = \int_0^{2\pi}\left(\frac{1}{2}\right)\,d\theta = \pi,$$

as required.

For the second part, note that

$$X = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy,$$

since all that changes is the dummy variable of integration. But then, by the solution to Problem 1,

$$\pi = \int_{R^2} e^{-x^2-y^2}\,d(x,y) = \int_{R^2} e^{-x^2}e^{-y^2}\,d(x,y) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = X^2.$$

But $X \geq 0$, since it is the integral of a nonnegative function, so we conclude that $X = \sqrt{\pi}$. ■

Problem 10: Let V_n denote the n-measure of the n-ball $B(0;1)$ of radius 1. Prove that $V_n = \pi^{n/2}/\Gamma(\frac{1}{2}n+1)$, where

$$\Gamma(s+1) \stackrel{\text{def}}{=} \int_0^\infty t^s e^{-t}\,dt$$

exists as an improper Riemann integral for all $s > -1$.

Solution 10: We prove this by induction on $n \geq 1$. We will exploit the identity $\Gamma(s+1) = s\Gamma(s)$ for all $s > 0$, which may be proved using integration by parts. Since a direct evaluation of the integral when $s = 0$ gives $\Gamma(1) = 1$, we get the identity $\Gamma(s+1) = s!$ whenever s is a nonnegative integer.

To begin the induction, note that $V_1 = 2$ since the 1-ball is an interval. Now,

$$\Gamma(\frac{1}{2}+1) = \int_0^\infty t^{1/2} e^{-t}\, dt = 2\int_0^\infty s^2 e^{-s^2}\, ds = -\int_{s=0}^\infty s\, de^{-s^2} = \int_{s=0}^\infty e^{-s^2}\, ds = \frac{1}{2}\pi^{1/2},$$

where we applied the coordinate transformation variable $t \leftarrow s^2$, performed integration by parts, and used Solution 9. That verifies the formula $V_1 = \pi^{1/2}/\Gamma(\frac{1}{2}+1)$ for the case $n = 1$.

Second, note that $V_2 = \pi$ by elementary geometry, and since $\Gamma(\frac{2}{2}+1) = \Gamma(1+1) = 1! = 1$, we have $\pi^{2/2}/\Gamma(\frac{2}{2}+1) = \pi$ as well. That verifies the case $n = 1$.

We next note that if $r > 0$ and $V_n(r)$ is the volume of the n-ball $B(0; r)$, then by the coordinate transformation $\mathbf{x} \leftarrow r\mathbf{y}$ we obtain the identity $V_n(r) = r^n V_n(1) = r^n V_n$. When needed, we will write $B_n(r)$ for the n-ball of radius r, centered at 0.

Now suppose that $n \geq 3$, to obtain a recursive formula for V_n in terms of V_{n-2}. For any $0 < r < 1$, we separate the coordinates $(x_1, \ldots, x_n)$ of a point in the unit n-ball into the pair $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, satisfying $x_1^2 + x_2^2 = r^2$, and the remainder $(x_3, \ldots, x_n)$, satisfying $x_3^2 + \cdots + x_n^2 = 1 - r^2$. Using Fubini's theorem and polar coordinates, we obtain:

$$V_n = \int_{\theta=0}^{2\pi} \left[\int_{r=0}^1 \left(\int_{B_{n-2}(\sqrt{1-r^2})} 1\, d(x_3, \ldots, x_n) \right) r dr \right] d\theta = 2\pi V_{n-2} \int_{r=0}^1 r\left(1-r^2\right)^{\frac{n}{2}-1} dr$$

It remains to evaluate $\int_{r=0}^1 r\left(1-r^2\right)^{\frac{n}{2}-1} dr = 1/n$, which is evident after the substitution $r \leftarrow \sqrt{t}$. Thus, by the inductive hypothesis,

$$V_n = \frac{2\pi}{n} V_{n-2} = \frac{\pi}{(n/2)} \frac{\pi^{n/2-1}}{\Gamma(\frac{n}{2})} = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2}+1)},$$

finishing the proof. ■