1. Suppose $f$ is analytic on the closed unit disk, $f(0) = 0$, and $|f(z)| \leq |e^z|$ whenever $|z| = 1$. How big can $f((1 + i)/2)$ be?

2. Prove Schwarz’s lemma for a disk of radius $R$: If $f$ is analytic on a closed disk $D$ of radius $R$ centered at $z_0$, $f(z_0) = 0$, and $|f(z)| \leq M$ on the boundary circle of $D$, then $|f(z)| \leq |z - z_0|M/R$ for each $z$ inside $D$, with equality holding at some interior point $z$ if and only if $f(z) = e^{ic}(z - z_0)$ for some constant $c \in \mathbb{R}$.

3. Use the radius-$R$ Schwarz lemma of Problem 2 to prove Liouville’s theorem. (Hint: apply the lemma to $f(z) - f(0)$.)

4. Prove that an entire function whose real part is bounded must be constant. (Hint: apply Liouville’s theorem to the function $e^f$.)

5. Suppose that $f$ is analytic on the closed unit disk, $f(0) = 0$, and $|\Re f(z)| \leq |e^z|$ for $|z| < 1$. Can $f((1 + i)/2)$ be $18$?