Ma 416: Complex Variables
Homework Assignment 11

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Due Thursday, December 1st, 2005

Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 4, sections 19A–20F.

1. Find an analytic function $f$ whose real part is $\Re f(x + iy) = x^3 y - xy^3$.

2. Find the conjugate harmonic function of $g(x, y) = e^{-y} \cos x$.

3. Is the function $h(x, y) = x^2 + y^2$ the imaginary part of some function $f(x + iy)$ analytic in the unit disk in $\mathbb{C}$?

4. For $0 \leq r < 1$ and $0 \leq \phi \leq 2\pi$, define the function

$$I(r, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) d\theta}{1 + r^2 - 2r \cos(\theta - \phi)}.$$  

(This is the integral of the Poisson kernel $P(r, \theta - \phi)$.)

(a) Show that $I(r, \phi)$ does not depend on $\phi$. (Hint: substitute $\theta \leftarrow \theta' + \phi$.)

(b) Show that $I(r, \phi)$ does not depend on $r$. (Hint: put $a = 2r/(1 + r^2)$, observe that $(1 - r^2)/(1 + r^2) = \sqrt{1 - a^2}$, and look up $\int_{-\pi}^{\pi} d\theta/(1 - a \cos \theta) = 2\pi/\sqrt{1 - a^2}$ in a table of integrals.)

(c) Show that $I(r, \phi) = 1$ for all $0 \leq r < 1$ and $0 \leq \phi \leq 2\pi$ by evaluating $I(0, 0) = 1$ and using parts (a) and (b).

(d) Conclude that if $u = u(x, y)$ is a harmonic function on the unit disk $D = \{x^2 + y^2 \leq 1\}$, and $u(x, y) = K$ for all $x^2 + y^2 = 1$, then $u(x, y) = K$ for all $(x, y) \in D$.

5. Show directly that $u(x, y) = x^2 - y^2$ satisfies the averaging property: if $R > 0$, $C_R = \{r(\theta) = (x_0 + R \cos \theta, y_0 + R \sin \theta) : 0 \leq \theta \leq 2\pi\}$, and $ds = \|r'(\theta)\| d\theta$ is the arc length differential on $C_R$, then

$$\oint_{C_R} u(x, y) \, ds = 2\pi R u(x_0, y_0).$$

How can Cauchy’s integral formula be used to derive the same results?