
1. Suppose that $u = u(x, y)$ is continuous on the closed unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$ and $u$ is twice continuously differentiable with $\Delta u(x, y) = 0$ inside $D$. Find a series representing $u$ for each of the following boundary conditions $u(\cos \theta, \sin \theta) = \psi(\theta)$, $-\pi \leq \theta \leq \pi$:

   (a) $\psi(\theta) = \begin{cases} 1, & \text{if } \theta \in [-\pi/2, \pi/2]; \\ 0, & \text{otherwise}, \end{cases}$

   (b) $\psi(\theta) = \sin \theta.$

2. Suppose that $u(x, y)$ is continuous on the closed annulus $A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ and $u$ is twice continuously differentiable with $\Delta u(x, y) = 0$ inside $A$. Find a series representing $u$ for each of the following boundary conditions $u(\cos \theta, \sin \theta) = \psi_1(\theta)$ and $u(2 \cos \theta, 2 \sin \theta) = \psi_2(\theta)$, $-\pi \leq \theta \leq \pi$:

   (a) $\psi_1(\theta) = 0; \quad \psi_2(\theta) = |\theta|;$

   (b) $\psi_1(\theta) = \cos \theta; \quad \psi_2(\theta) = \sin \theta.$

3. Suppose that $f = f(z)$ is analytic and univalent in a region $D \subset \mathbb{C}$ and let $E = f(D) = \{f(z) : z \in D\}$ be its range. Write $u + iv = f(x + iy)$ and identify $(u, v)$ with $u + iv$. Prove that if $\phi = \phi(u, v)$ is a harmonic function in $E$, then $\psi(x, y) \overset{\text{def}}{=} \phi(f(x + iy))$ is a harmonic function in $D$.

4. Find a Möbius transform mapping $0, 1, i$ to $\infty, 1, -i$, respectively. Is it unique?

5. Find all the Möbius transforms that the unit disk $\{|z| < 1\}$ to its exterior $\{|z| > 1\}$. 