1. Prove or find a counterexample to the following statements:
(a) If \( f(x) = O(g(x)) \) as \( x \to 0 \), then \( f(x)/g(x) \to 0 \) as \( x \to 0 \).
(b) If \( f(x) = o(g(x)) \) as \( x \to \infty \), then \( f(x)/|1 + g(x)| \to 0 \) as \( x \to \infty \).
(c) If \( f(x) = o(g(x)) \) as \( x \to 1 \), then \( f(x) = O(g(x)) \) as \( x \to 1 \).
(d) If \( f(x) = o(x) \) as \( x \to 0 \), then \( f(x) = O(x^2) \) as \( x \to 0 \).

**Solution:** (a) This is false. Let \( f(x) = g(x) = 1 \) for all \( x \); then \( f(x)/g(x) = 1 \) for all \( x \) and cannot have 0 as a limit as \( x \to 0 \).
(b) This is true. By definition, \( f(x) = o(g(x)) \) as \( x \to \infty \) implies that \( f(x)/g(x) \to 0 \) as \( x \to \infty \). But then \( |f(x)/g(x)| \to 0 \) as \( x \to \infty \), and

\[
0 \leq \frac{|f(x)|}{1 + |g(x)|} \leq \frac{|f(x)|}{|g(x)|} \to 0, \quad \text{as } x \to \infty,
\]

so the result follows from the squeeze law of limits.
(c) This is true. The hypothesis \( f(x) = o(g(x)) \) as \( x \to \infty \) implies that \( |f(x)/g(x)| \to 0 \) as \( x \to \infty \). But then for any \( \epsilon > 0 \) there must be some \( \delta > 0 \) such that \( |f(x)/g(x)| \leq \epsilon \) for all \( x \) satisfying \( |x - 1| < \delta \). Choosing \( \epsilon = 2 \) and finding the corresponding \( \delta \) yields the result:

\[
|f(x)| \leq 2|g(x)| \quad \text{for all } x \text{ with } |x - 1| < \delta,
\]

which is a particular case of the statement \( f(x) = O(g(x)) \) as \( x \to 1 \).
(d) This is false. The function \( f(x) = x\sqrt{|x|} \) satisfies \( f(x) = o(x) \) as \( x \to 0 \) but not \( f(x) = O(x^2) \) as \( x \to 0 \), since \( |f(x)/x^2| = 1/\sqrt{|x|} \) is not bounded in any neighborhood of \( x = 0 \). \( \Box \)

2. Let \( f(x, y) = u(x, y) + iv(x, y) \) be a complex-valued function of two real variables. Write \( z = x + iy \) for the complex variable with real part \( x \) and imaginary part \( y \). Show that the Cauchy-Riemann equations are equivalent to the equation

\[
\frac{\partial}{\partial z} f(z) = 0,
\]

using the definition \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{i \partial}{\partial y} \right) \) on page 19 of our textbook.

**Solution:** Compute

\[
\frac{\partial}{\partial z} f(z) = \frac{1}{2} \left[ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right]
\]

If \( \frac{\partial}{\partial z} f(z) = 0 \), then both the real and imaginary parts of the derivative must be zero, so

\[
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0; \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.
\]
These are the Cauchy-Riemann equations. Conversely, if the Cauchy-Riemann equations are satisfied by $u$ and $v$, then we will have $\frac{\partial}{\partial z} f(z) = 0$ for $f = u + iv$.

Note that the factor $\frac{1}{2}$ in the definition of $\frac{\partial}{\partial z}$ plays no role in this equivalence.

3. Determine whether the following functions $f(z) = f(x + iy)$ are analytic:
   (a) $f(z) = x^2 + y^2$
   (b) $f(z) = x^2 - y^2$
   (c) $f(z) = x^2 - y^2 + 2i xy$

**Solution:**
   (a) No. We may write $f(z) = |z|^2 = z \bar{z}$, so $\frac{\partial}{\partial z} f(z) = z \neq 0$.
   (b) No. Write $u(x, y) = x^2 - y^2$ and $v(x, y) = 0$. Then the Cauchy-Riemann equations are not satisfied, since $\frac{\partial u}{\partial x} = 2x \neq 0 = \frac{\partial v}{\partial y}$.
   (c) Yes, as we may write $f(z) = z^2$ which satisfies $\frac{\partial}{\partial z} f(z) = 0$ for all $z$. Hence the Cauchy-Riemann equations are satisfied. But also, the real and imaginary parts of $f$ are continuous and have continuous partial derivatives (as they are polynomials), so by exercise 2.3 on page 17 of the text, $f$ is analytic.

4. Find the domain of convergence of the following power series:
   (a) $\sum_{n=0}^{\infty} (z - 3i)^{2n}$
   (b) $\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}$

**Solution:**
   (a) By the ratio test, the radius of convergence is 1 about the point $3i$, so the domain of convergence is $\{z : |z - 3i| < 1\}$.
   (b) By the ratio test, the radius of convergence is $\infty$, so the domain of convergence is the entire complex plane.

5. Write a power series for the $k^{th}$ derivative of
   $\sum_{n=0}^{\infty} (-1)^n z^n$,
   for all $k = 1, 2, \ldots$, and determine the domain of convergence. What functions do these power series represent?

**Solution:**
   The power series about $z = 0$ for the $k^{th}$ derivative is
   $\sum_{n=k}^{\infty} (-1)^n (n) \cdots (n-k+1) z^{n-k} = \sum_{n=k}^{\infty} (-1)^n \frac{n!}{(n-k)!} z^{n-k}$,
   for all $k = 1, 2, \ldots$. The domain of convergence is $\{|z| < 1\}$ in all cases. These power series represent the functions
   $\frac{d^k}{dz^k} \left[ \frac{1}{1+z} \right] = \frac{(-1)^k k!}{(1+z)^{k+1}}$
on the domain of convergence.

6. Determine, with proof, whether the following series converge uniformly on the domain $|z| < 1$:
   (a) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$
   (b) $\sum_{n=0}^{\infty} z^n$.
Solution: (a) Yes. This series satisfies the Weierstrass $M$-test with constants $M_n = 1/n^2$, and everyone knows that $\sum 1/n^2 = \pi^2/6 < \infty$.

(b) No. The series diverges at $z = 1$, suggesting nonuniform convergence near there. For proof, for any real $0 < z < 1$ and any two integers $0 < P < Q$ we may compute

$$\left| \sum_{n=P}^{Q-1} z^n \right| = \sum_{n=P}^{Q-1} z^n = \sum_{n=0}^{Q-1} z^n - \sum_{n=0}^{P-1} z^n = \frac{z^P - z^Q}{1-z}.$$  

To have uniform convergence, it is necessary that the rightmost expression can be made arbitrarily small for all $|z| < 1$ and any $Q > P$ simply by choosing large enough $P$. However, given any fixed $P$ we observe that

$$\lim_{z \to 1^-} \left( \lim_{Q \to \infty} \frac{z^P - z^Q}{1-z} \right) = \lim_{z \to 1^-} \frac{z^P}{1-z} = +\infty,$$

so for every $P$ there will always be some combination of $z$ near 1 and big $Q$ that yields big $\left| \sum_{n=P}^{Q-1} z^n \right|$. Hence the convergence of $\sum z^n$ cannot be uniform on $[0,1)$. \qed