Ma 416: Complex Variables
Solutions to Homework Assignment 3

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Due Thursday, September 22nd, 2005

1. Find the Maclaurin series of $\sinh z = \frac{1}{2}(e^z - e^{-z})$.

   Solution: The even-power terms of the exponential series cancel, leaving

   $$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

   □

2. Show that $\sinh z$ has infinitely many zeroes. (Hint: first express $\sinh z$ in terms of the sine function.)

   Solution: Follow the hint. Since $e^{iz} = \cos z + i \sin z$, compute

   $$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \quad \Rightarrow \sin iz = \frac{1}{2i}(e^{-z} - e^{z}) = -\frac{1}{i}\sinh z = i \sinh z.$$

   Thus $\sinh z = -i \sin iz$. This means $\sinh z = 0$ for every $z = ik\pi$ with $k \in \mathbf{Z}$. □

3. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges for every $x \in [0,1]$, prove that $\sum_{n=1}^{\infty} a_n x^{n-1}$ converges in the same interval.

   Solution: Since all terms are nonnegative, we may employ the comparison test. For any $0 < P < Q$, write

   $$0 \leq \sum_{n=P}^{Q} a_n x^n = x \sum_{n=P}^{Q} a_n x^{n-1} \leq x \sum_{n=P}^{Q} n a_n x^{n-1}.$$

   But since $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges for every $x \in [0,1]$, for every $\epsilon > 0$ we can find sufficiently large P so that $0 \leq \sum_{n=P}^{Q} n a_n x^{n-1} < \epsilon$ for any $Q > P$. Hence for every $x \in [0,1]$ and every $\epsilon > 0$ we can find sufficiently large P so that $0 \leq \sum_{n=P}^{Q} a_n x^n < \epsilon$ for any $Q > P$. By definition, therefore, $\sum_{n=0}^{\infty} a_n x^n$ converges for each $x \in [0,1]$. □

4. Suppose that an analytic function f has arbitrarily small periods. That is, suppose that there is an infinite sequence $\{p_k : k \in \mathbf{N}\}$ with $|p_k| \to 0$ as $k \to \infty$ such that $f(z + p_k) = f(z)$ for all k and all $z \in \mathbf{C}$. Prove that f must be constant.

   Solution: Fix an arbitrary $z \in \mathbf{C}$ and observe that if $f(z + p_k) = f(z)$ for all k, then $f'(z) = \lim_{k\to\infty}(f(z + p_k) - f(z))/p_k = 0$. Since z was arbitrary, we have found that $f'(z) = 0$ in all of $\mathbf{C}$. Thus f must be constant. □

5. (a) Is there a solution $z \in \mathbf{C}$ to the equation $e^z = 0$? (b) Is there a solution $z \in \mathbf{C}$ to the equation $\tan z = i$?

   Solution: (a) No such solution exists, for if it did it would imply that $e^w = e^{w-z}e^z = e^{w-z}0 = 0$ for every complex number w, contradicting $e^0 = 1 \neq 0$.

   (b) No such solution exists. Use Euler's formula to write

$$\tan z = \frac{\sin z}{\cos z} = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

   Thus $\tan z = i$ implies that $e^{iz} - e^{-iz} = -[e^{iz} - e^{-iz}]$, so $e^{iz} = 0$. But by part (a), this has no solution. □

6. Obtain formulas for the sums $\sin\theta + \sin 2\theta + \cdots + \sin n\theta$ and $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$ by considering the real and imaginary parts of the geometric series $\sum_{k=0}^{n} e^{ik\theta}$.

   Solution: De Moivre's formulas yield

$$\sum_{k=0}^{n} e^{ik\theta} = [1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta] + i[\sin\theta + \sin 2\theta + \cdots + \sin n\theta]$$

   Alternatively, the goemetric sum formula yields

$$\begin{aligned}
\sum_{k=0}^{n} e^{ik\theta} &= \sum_{k=0}^{n} (e^{i\theta})^k = \frac{1 - e^{i[n+1]\theta}}{1 - e^{i\theta}} \\
&= \left(\frac{e^{i\frac{(n+1)\theta}{2}}}{e^{i\frac{\theta}{2}}}\right)\left(\frac{e^{-i\frac{[n+1]\theta}{2}} - e^{i\frac{[n+1]\theta}{2}}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}}\right) = e^{i\frac{n\theta}{2}}\left(\frac{\sin[n+1]\theta/2}{\sin\theta/2}\right)
\end{aligned}$$

   Separating the real and imaginary parts gives

$$\begin{aligned}
1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta &= \left(\frac{\sin[n+1]\theta/2}{\sin\theta/2}\right)\cos\frac{n\theta}{2} \\
\sin\theta + \sin 2\theta + \cdots + \sin n\theta &= \left(\frac{\sin[n+1]\theta/2}{\sin\theta/2}\right)\sin\frac{n\theta}{2}
\end{aligned}$$

□

7. Let $C = \{z : |z| = r\}$ be a circle of radius $r > 0$, centered at the origin in $\mathbf{C}$, equipped with the positive (counterclockwise) orientation. (a) Compute $\int_C (1/z)\, dz$. (b) Compute $\int_C (1/\bar{z})\, dz$. (Hint: parametrize C.)

Solution: Following the hint, write $C = \{re^{it} : 0 \le t < 2\pi\}$.

(a)
$$\int_C \frac{1}{z}\, dz = \int_{t=0}^{2\pi} (re^{it})^{-1} re^{it} i\, dt = i \int_{t=0}^{2\pi} dt = 2\pi i.$$
Note that this is independent of r.

(b)
$$\int_C \frac{1}{\bar{z}}\, dz = \int_{t=0}^{2\pi} (re^{-it})^{-1} re^{it} i\, dt = i \int_{t=0}^{2\pi} e^{2it}\, dt = 0.$$
Note that this too is independent of r. □

8. Find all the zeros of the function $f(z) = 2 + \cos z$. (Hint: if they exist, they must be nonreal.)

Solution: Following the hint, write $z = x + iy$ with real and imaginary parts $x, y \in \mathbf{R}$. But then

$$\cos z = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y,$$

since $\cos iy = \cosh y$ and $\sin iy = i \sinh y$. To solve $2 + \cos z = 0$ is thus equivalent to finding $z = x + iy$ such that $\cos x \cosh y = -2$ and $\sin x \sinh y = 0$.

Now $\sin x \sinh y = 0$ if and only if either $\sinh y = 0$ or $\sin x = 0$. The first case is excluded because it requires $y = 0$, so $\cosh y = 1$, so $\cos x = -2$ which cannot happen. The second case is equivalent to $x = k\pi$ for $k \in \mathbf{Z}$. Now $\cosh y = \frac{1}{2}(e^y + e^{-y}) \ge 1$ for all real y with equality if and only if $y = 0$; otherwise, $\cosh y = C$ has two distinct real roots for every $C > 1$. We conclude that

$$-2 = \cos x \cosh y = \cos k\pi \cosh y = (-1)^k \cosh y$$

has a solution if and only if $x = k\pi$ for some odd integer k and y is one of the two real roots of $\cosh y = 2$. □