Ma 416: Complex Variables
Solutions to Homework Assignment 4

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Due Thursday, September 29th, 2005

1. Let $f_n(x) = [x^n(1-x^n)]$ for $n = 1, 2, 3, \ldots$. Does the sequence $\{f_n(x)\}$ converge uniformly on $0 < x < 1$?

   Solution: No. For any fixed $x \in (0,1)$, $f_n(x) \to 0$ as $n \to \infty$ by the squeeze lemma:

   $$\lim_{n\to\infty} x^n = 0; \qquad \lim_{n\to\infty}(1-x^n) = 1; \qquad \Rightarrow 0 \le \lim_{n\to\infty} f_n(x) = 0.$$

   However, $\sqrt[n]{\frac{1}{2}} \in (0,1)$ and $f_n(\sqrt[n]{\frac{1}{2}}) = \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$ for any n, so for $0 < \epsilon < \frac{1}{4}$ it is impossible to specify N large enough to guarantee

   $$n < N \Rightarrow (\forall x \in (0,1))\, |f_n(x) - 0| < \epsilon,$$

   since $|f_n(x) - 0| = \frac{1}{4}$ for some x no matter what n is. □

2. Use Cauchy's Inequalities to deduce Liouville's Theorem.

   Solution: Assume Cauchy's Inequalities and suppose that f is a bounded function analytic on $\mathbf{C}$. Then there is some $M < \infty$ satisfying $|f(z)| \le M$ for all $z \in \mathbf{C}$. Let a_n be the n^{th} Taylor coefficient of f expanded about $z_0 = 0$:

   $$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

   For each $n \ge 1$ we show that $a_n = 0$ by showing that $|a_n| < \epsilon$ for every $\epsilon > 0$. So fix $n \ge 1$ and let $\epsilon > 0$ be given. Take $r > 0$ large enough so that $Mr^{-n} < \epsilon$. Since f is analytic on $\mathbf{C}$, it is analytic on $D_r \stackrel{\text{def}}{=} \{|z| \le r\}$, and since M bounds $|f|$ on $\mathbf{C}$ we have $|f(z)| \le M$ on $D_r \subset \mathbf{C}$. By Cauchy's Inequality for a_n we may conclude that $|a_n| \le Mr^{-n} < \epsilon$. Thus $a_n = 0$ for $n \ge 1$, so $f(z) = a_0$.

   But this prove that a bounded function analytic on $\mathbf{C}$ must be a constant. □

3. Let $D \subset \mathbf{C}$ be the closed diamond-shaped region with vertices $1, i, -1, -i$. Suppose that $f = f(z)$ is analytic on D and satisfies $|f(z)| \le M$ for all $z \in D$. Prove that $|f'(0)| \le M\sqrt{2}$ and $|f''(0)| \le 4M$.

Solution: Let C be the largest circle centered at 0 that fits inside D. Then C has radius $1/\sqrt{2}$. We use Cauchy's formulas to estimate the derivatives:

$$|f'(0)| = \left| \frac{1!}{2\pi i} \int_C \frac{f(w)}{(w-0)^2} \, dw \right| \leq \frac{M}{2\pi} \int_C \frac{dw}{|w|^2} = \frac{M}{2\pi} \frac{2\pi(1/\sqrt{2})}{(1/\sqrt{2})^2} = M\sqrt{2}.$$

$$|f''(0)| = \left| \frac{2!}{2\pi i} \int_C \frac{f(w)}{(w-0)^3} \, dw \right| \leq \frac{2M}{2\pi} \int_C \frac{dw}{|w|^3} = \frac{2M}{2\pi} \frac{2\pi(1/\sqrt{2})}{(1/\sqrt{2})^3} = 4M.$$

Using a smaller circle for C would give a larger right-hand side, and thus a weaker estimate, in both cases. □

4. Suppose that $f(z)$ is analytic on $|z| < 2$. Define $F_0(z) = f(z)$ and $F_{n+1}(z) = \int_0^z F_n(w) \, dw$ for $n \geq 0$. Prove that if $\{F_n(z)\}$ converges uniformly on $|z| < 1$, then $f(z) = ce^z$ for some constant c.

   Solution: Let $g(z) = \lim_{n\to\infty} F_n(z)$. Since $F_n'(z) = F_{n-1}(z)$, we also have $g'(z) = \lim_{n\to\infty} F_n'(z) = g(z)$. We conclude that $g(z) = ce^z$ for some constant c. □

5. Recall that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for all real x. Show that

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z,$$

   for all complex z. (Hint: use the uniform convergence theorem and the coincidence principle.)

   Solution: Following the hint, define $f_n(z) \stackrel{\text{def}}{=} \left(1 + \frac{z}{n}\right)^n - e^z$ for $n = 1, 2, \ldots$. This sequence of entire analytic functions converges uniformly on the curve defined by the real interval $[1, 0] \subset \mathbf{R} \subset \mathbf{C}$. (Any other positive-length bounded interval in $\mathbf{R}$ would work equally well.) By the uniform convergence theorem on page 58 of our text, $f \stackrel{\text{def}}{=} \lim_{n\to\infty} f_n$ is an entire analytic function. But $f(z) = 0$ for all real $z \in [0, 1]$, so $f(z) = 0$ for all $z \in \mathbf{C}$ by the coincidence principle on page 63 of our text. □

6. Compute $\Gamma(3/2)$ and $\Gamma(-1/2)$.

   Solution: Use the identity $\Gamma(z + 1) = z\Gamma(z)$ with the fact that $\Gamma(1/2) = \sqrt{\pi}$ to compute $\Gamma(3/2) = \Gamma(1 + 1/2) = \frac{1}{2}\sqrt{\pi}$.
   Likewise $(-1/2)\Gamma(-1/2) = \Gamma(1 - 1/2) = \Gamma(1/2) = \sqrt{\pi}$, so $\Gamma(-1/2) = -2\sqrt{\pi}$. □