

Ma 416: Complex Variables

Solutions to Homework Assignment 6

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Read R. P. Boas, *Invitation to Complex Analysis*, Chapter 2, sections 9A–11C.

1. Evaluate the definite integral $\int_0^{2\pi} (2 + \sin \theta)^{-2} d\theta$.

Solution: Substitute $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta \Rightarrow d\theta = (1/iz) dz$, and $\sin \theta = (z - 1/z)/2i$ to get

$$\int_0^{2\pi} \frac{d\theta}{(2 + \sin \theta)^2} = \oint_C \frac{dz}{iz \left(2 + \frac{z-1/z}{2i}\right)^2} = \oint_C \frac{4iz dz}{(z^2 + 4iz - 1)^2} = \oint_C \frac{4iz dz}{(z - z_+)^2(z - z_-)^2},$$

where C is the positively oriented unit circle $\{z = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, and $z_{\pm} = (-2 \pm \sqrt{3})i$ are the roots of the denominator quadratic polynomial.

Since $z_+ = (-2 + \sqrt{3})i$ lies in the region $\{|z| < 1\}$ enclosed by C , it is a pole of order 2 for the integrand $f(z) \stackrel{\text{def}}{=} 4iz(z - z_+)^{-2}(z - z_-)^{-2}$. The enclosed region contains no other poles of f , since $|z_-| = 2 + \sqrt{3} > 1$. By the residue theorem, the integral equals $2\pi i \text{Res}(z_+)$, where $\text{Res}(z_+)$ is the residue of f at z_+ . Evaluate this residue with Cauchy's formula on page 73 of our text, with $n = 2$:

$$\begin{aligned} \text{Res}(z_+) &= \lim_{z \rightarrow z_+} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{(n-1)} [f(z)(z - z_+)^n] = \lim_{z \rightarrow z_+} \frac{d}{dz} \left[\frac{4iz}{(z - z_-)^2} \right] \\ &= \lim_{z \rightarrow z_+} \frac{-4i(z + z_-)}{(z - z_-)^3} = \frac{-4i(z_+ + z_-)}{(z_+ - z_-)^3} = \frac{2}{3\sqrt{3}i}. \end{aligned}$$

Hence $2\pi i \text{Res}(z_+) = \frac{4\pi}{3\sqrt{3}}$ is the value of the original integral. □

2. Evaluate the improper integral $\int_0^{\infty} (x^4 + 1)^{-1} dx$.

Solution: Write $z^4 + 1 = (z^2 + i)(z^2 - i) = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$, where $z_1 = (1 + i)/\sqrt{2}$, $z_2 = (-1 + i)/\sqrt{2}$, $z_3 = (-1 - i)/\sqrt{2}$, and $z_4 = (1 - i)/\sqrt{2}$ are the four 4th roots of -1 .

Next, fix big $R > \sqrt{2}$, let $C = C_h \cup C_a \cup C_v$ be the simple closed piecewise smooth quarter-circle contour in the first quadrant consisting of the horizontal line segment $C_h = \{z = x : x \in [0, R]\}$, the quarter-arc $C_a = \{Re^{i\theta} : 0 \leq \theta \leq \pi/2\}$, and the vertical line segment $C_v = \{z = (R - y)i : y \in [0, R]\}$. As parametrized, C has positive orientation.

The region enclosed by C will contain the simple pole z_1 of $f(z) \stackrel{\text{def}}{=} (z^4 + 1)^{-1}$ and no other singular points of f . Hence by the residue theorem, $\oint_C f(z) dz = 2\pi i \text{Res}(z_1)$, where $\text{Res}(z_1)$ is the residue of f at z_1 . But since z_1 is a simple pole,

$$\text{Res}(z_1) = \lim_{z \rightarrow z_1} (z - z_1)f(z) = [(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)]^{-1} = \frac{1}{2i(1+i)\sqrt{2}}.$$

Thus $\oint_C f(z) dz = \frac{\pi}{(1+i)\sqrt{2}}$ for any $R > \sqrt{2}$.

Also, $\oint_C f(z) dz = \int_{C_h} f(z) dz + \int_{C_a} f(z) dz + \oint_{C_v} f(z) dz$, where

$$\begin{aligned} \oint_{C_h} f(z) dz &= \int_0^R \frac{dx}{x^4 + 1}; \\ \oint_{C_v} f(z) dz &= \int_0^R \frac{di(R-y)}{(R-y)^4 + 1} = -i \int_0^R \frac{dt}{t^4 + 1}, \quad \text{after } y \leftarrow (R-t); \\ \oint_{C_a} f(z) dz &= \int_0^{\pi/2} \frac{iRe^{i\theta} d\theta}{R^4 e^{4i\theta} + 1} \\ &\Rightarrow \left| \oint_{C_a} f(z) dz \right| \leq \frac{\pi}{2} \frac{R}{R^4 - 1} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \end{aligned}$$

The integral over the arc C_a must therefore vanish as $R \rightarrow \infty$, leaving

$$\lim_{R \rightarrow \infty} \oint_C f(z) dz = (1-i) \int_0^\infty \frac{dx}{x^4 + 1}.$$

The limit $\pi/[(1+i)\sqrt{2}]$ on the left-hand side is attained as soon as $R > \sqrt{2}$. Hence

$$\int_0^\infty \frac{dx}{x^4 + 1} = \frac{\pi}{(1-i)(1+i)\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

□

3. Evaluate the principal value integral $\text{PV} \int_{-\infty}^\infty (x^3 - 1)^{-1} dx$.

Solution: Note that $f(z) \stackrel{\text{def}}{=} (z^3 - 1)^{-1}$ has simple poles at the three cube roots of 1, namely $z_1 = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$, $z_2 = e^{4\pi i/3} = (-1 - i\sqrt{3})/2$, and $z_3 = e^{6\pi i/3} = 1$.

Fix $R > 2$ and $0 < \epsilon < 1$ and consider the piecewise differentiable curve $C = C_1 \cup C_\epsilon \cup C_2 \cup C_R$ defined by

$$\begin{aligned} C_1 &= \{z = x : -R \leq x \leq 1 - \epsilon\}; \\ C_\epsilon &= \{z = 1 - \epsilon e^{-i\theta} : 0 \leq \theta \leq \pi\}; \\ C_2 &= \{z = x : 1 + \epsilon \leq x \leq R\}; \\ C_R &= \{z = R e^{i\theta} : 0 \leq \theta \leq \pi\}. \end{aligned}$$

Then C is a simple closed curve that encloses one pole of f (namely z_1), skirts within ϵ of another ($z_3 = 1$), and completely excludes the third (z_2). Note that C is positively oriented. By the residue theorem,

$$\oint_C f(z) dz = 2\pi i \text{Res}(z_1) = 2\pi i \lim_{z \rightarrow z_1} (z - z_1)f(z) = \frac{2\pi i}{(z_1 - z_2)(z_1 - z_3)} = \frac{4\pi}{3(i - \sqrt{3})}.$$

The four path integrals comprising $\oint_C f(z) dz$ may be written as follows:

$$\begin{aligned}\int_{C_1 \cup C_2} f(z) dz &= \int_{-R}^{1-\epsilon} \frac{dx}{x^3-1} + \int_{1+\epsilon}^R \frac{dx}{x^3-1}; \\ \int_{C_\epsilon} f(z) dz &= \int_0^\pi \frac{i\epsilon e^{-i\theta} d\theta}{[1 - \epsilon e^{-i\theta}]^3 - 1} = \int_0^\pi \frac{i d\theta}{-3 + 3\epsilon e^{-i\theta} - \epsilon^2 e^{-2i\theta}}; \\ \int_{C_R} f(z) dz &= \int_0^\pi \frac{iR e^{i\theta} d\theta}{R^3 e^{3i\theta} - 1}.\end{aligned}$$

Evidently $\int_{C_\epsilon} f(z) dz \rightarrow -i\pi/3$ as $\epsilon \rightarrow 0$, and $\int_{C_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$.

As $\epsilon \rightarrow 0$ and $R \rightarrow \infty$, the path integral $\int_{C_1 \cup C_2} f(z) dz$ converges to the principal value we are seeking. Combining these calculations gives

$$\text{PV} \int_{-\infty}^{\infty} (x^3 - 1)^{-1} dx = \frac{4\pi}{3(i - \sqrt{3})} - \frac{-i\pi}{3} - 0 = \frac{-\pi}{\sqrt{3}}$$

Note that the negative values of the integrand in $-\infty < x < 1$ outweigh the positive values in $1 < x < \infty$. □

4. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{x/n}}{1 - e^x} dx,$$

for $n = 1, 2, 3, \dots$ (Hint: use rectangular contours of fixed height 2π with one side lying on the x -axis.)

Solution: Following the hint, fix $R > 2$ and $0 < \epsilon < 1$ and let $C = C_B \cup C_{\epsilon B} \cup C_U \cup C_T \cup C_{\epsilon T} \cup C_D$ be the piecewise differentiable rectangular contour with two nibbles near $z = 0$ and $z = 2\pi i$ defined as follows:

$$\begin{aligned}C_B &= \{z = x : -R \leq x \leq -\epsilon \text{ and } \epsilon \leq x \leq R\}; \\ C_T &= \{z = 2\pi i - x : -R \leq x \leq -\epsilon \text{ and } \epsilon \leq x \leq R\}; \\ C_U &= \{z = R + iy : 0 \leq y \leq 2\pi\}; \\ C_D &= \{z = -R + i(2\pi - y) : 0 \leq y \leq 2\pi\}; \\ C_{\epsilon B} &= \{z = -\epsilon e^{-i\theta} : 0 \leq \theta \leq \pi\}; \\ C_{\epsilon T} &= \{z = 2\pi i + \epsilon e^{-i\theta} : 0 \leq \theta \leq \pi\}.\end{aligned}$$

It may be seen that C is a simple closed curve.

For each integer $n > 1$, the function $f(z) = e^{z/n}/(1 - e^z)$ only has simple poles at $z_k = 2k\pi i$ for any integer k , so it has no singular points in the region enclosed by C . The nibbles $C_{\epsilon B}$ and $C_{\epsilon T}$ avoid the poles $z_0 = 0$ and $z_1 = 2\pi i$, respectively. By the residue theorem, $\oint_C f(z) dz = 0$.

Now

$$\int_{C_B} f(z) dz = \int_{-R}^{-\epsilon} \frac{e^{x/n} dx}{1 - e^x} + \int_{\epsilon}^R \frac{e^{x/n} dx}{1 - e^x} \stackrel{\text{def}}{=} \int_I \frac{e^{x/n} dx}{1 - e^x},$$

where $I = [-R, -\epsilon] \cup [\epsilon, R]$. This tends to our desired integral as $R \rightarrow \infty$ and $\epsilon \rightarrow 0$. But also,

$$\int_{C_T} f(z) dz = \int_I \frac{e^{[2\pi i - x]/n} d[2\pi i - x]}{1 - e^{[2\pi i - x]}} = e^{2\pi i/n} \int_I \frac{e^{-x/n} d(-x)}{1 - e^{-x}} = -e^{2\pi i/n} \int_I \frac{e^{x/n} dx}{1 - e^x},$$

after the substitution $x \leftarrow -x$. Thus

$$\int_{C_B \cup C_T} f(z) dz = (1 - e^{2\pi i/n}) \int_I \frac{e^{x/n} dx}{1 - e^x}.$$

The values for $\int_{C_{\epsilon B}} f(z) dz$ and $\int_{C_{\epsilon T}} f(z) dz$ are computed as in Exercise 9.5 on pages 84–85 of our text. They are “partial residues,” exactly half the residues at z_0 and z_1 , respectively, and negative because the poles are outside the contour C :

$$\begin{aligned} \int_{C_{\epsilon B}} f(z) dz &= -\frac{1}{2}[2\pi i]\text{Res}(z_0) = \pi i; \\ \int_{C_{\epsilon T}} f(z) dz &= -\frac{1}{2}[2\pi i]\text{Res}(z_1) = e^{2\pi/n}\pi i; \end{aligned}$$

Here we have computed the residues in the usual way:

$$\begin{aligned} \text{Res}(z_0) &= \text{Res}(0) = \lim_{z \rightarrow 0} \frac{ze^{z/n}}{1 - e^z} = -1; \\ \text{Res}(z_1) &= \text{Res}(2\pi i) = \lim_{z \rightarrow 2\pi i} \frac{(z - 2\pi i)e^{z/n}}{1 - e^z} = -e^{2\pi i/n} \end{aligned}$$

The remaining contour integrals may be estimated:

$$\begin{aligned} \left| \int_{C_U} f(z) dz \right| &\leq \int_0^{2\pi} \frac{e^{R/n} |e^{iy/n}| dy}{|1 - e^R e^{iy}|} \leq \frac{2\pi e^{R/n}}{e^R - 1} \rightarrow 0, \quad \text{as } R \rightarrow \infty; \\ \left| \int_{C_D} f(z) dz \right| &\leq \int_{2\pi}^0 \frac{e^{-R/n} |e^{iy/n}| dy}{|1 - e^{-R} e^{iy}|} \leq \frac{2\pi e^{-R/n}}{1 - e^{-2}} \rightarrow 0, \quad \text{as } R \rightarrow \infty. \end{aligned}$$

Combining these computations gives the value of the integral:

$$\int_{-\infty}^{\infty} \frac{e^{x/n} dx}{1 - e^x} = \frac{-(1 + e^{2\pi i/n})\pi i}{1 - e^{2\pi i/n}} = \frac{(e^{-\pi i/n} + e^{\pi i/n})\pi i}{-(e^{-\pi i/n} - e^{\pi i/n})} = \pi \cot(\pi/n).$$

Note that this integral must be interpreted in the principal value sense. □

5. For each relation below, find all the complex numbers z satisfying it:

- (a) $z^n = 1$ for a fixed $n \in \{2, 3, 4, \dots\}$
- (b) $e^z = -e$
- (c) $e^{\sqrt{z}} = i$
- (d) $\tan z = 1 + i$.

Solution: (a) $z \in \{e^{2k\pi i/n} : k = 1, 2, \dots, n\}$. There are n distinct values in this set, the n n^{th} roots of unity, equidistributed around the unit circle.

(b) Since $-e = e^{i\pi} e^1 = e^{1+i\pi}$ and the exponential function is $2\pi i$ -periodic, the solution set is $\{1 + \pi i + 2k\pi i = 1 + (2k + 1)\pi i : k \in \mathbf{Z}\}$.

(c) Since $e^{i\pi/2} = i$ and the exponential function is $2\pi i$ -periodic, the solution set is $\{z^2 : z = i\pi/2 + 2k\pi i = (2k + \frac{1}{2})\pi i : k \in \mathbf{Z}\} = \{-(2k + \frac{1}{2})^2 \pi^2 : k \in \mathbf{Z}\}$.

(d) Use the idea in the solution to Exercise 10.11 on page 95 of our text:

$$\begin{aligned} 1 + i = \tan z &= \frac{\sin z}{\cos z} = \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \\ &\Rightarrow (2 - i)e^{iz} = ie^{-iz} \\ &\Rightarrow e^{2iz} = \frac{i}{2 - i} = -\frac{1}{5} + \frac{2}{5}i = \frac{1}{\sqrt{5}} \left(-\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i \right). \end{aligned}$$

The final expression on the right-hand side is in the form $re^{i\theta}$ with $r = 1/\sqrt{5}$ and $\theta \in \mathbf{R}$ satisfying $\cos \theta = -1/\sqrt{5}$, $\sin \theta = 2/\sqrt{5}$. From this we conclude that

$$\begin{aligned} \Re(2iz) &= -2\Im(z) = \ln(r) \approx -0.8047 \\ &\Rightarrow \Im(z) \approx 1.6094; \\ \Im(2iz) &= 2\Re(z) = \theta \in \{\arctan(-2) + \pi + 2k\pi : k \in \mathbf{Z}\} \\ &\Rightarrow \Re(z) \in \{1.0172 + k\pi : k \in \mathbf{Z}\}, \end{aligned}$$

using the approximation $\arctan(-2) \approx -1.1071$, the principal value of arctangent which lies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We must add π to this principal value to get an angle $\theta \approx 2.0344$ in the second quadrant satisfying $\sin \theta > 0$, $\cos \theta < 0$. Hence the solution set may be written approximately as $\{1.0172 + 1.6094i + k\pi : k \in \mathbf{Z}\}$. \square

6. Evaluate the integral

$$\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}}.$$

Solution: This is a special case of Exercise 11.1(a) on page 97 of the textbook:

$$\int_0^\infty \frac{x^\lambda dx}{a^2 + x^2}, \quad a = 1, \lambda = -\frac{1}{2}.$$

We follow the model solution on page 294 of the textbook, substituting $x^2 \leftarrow a^2t = t$ to obtain the equivalent integral

$$\frac{1}{2}a^{\lambda-1} \int_0^\infty \frac{t^{-1+(1+\lambda)/2} dt}{1+t} = \frac{1}{2} \int_0^\infty \frac{t^{-3/4} dt}{1+t}, = \frac{1}{2} \int_0^\infty \frac{t^{\frac{1}{4}-1} dt}{1+t},$$

which evaluates to

$$\frac{1}{2}a^{\lambda-1}I\left(\frac{1}{4}\right) = \frac{1}{2} \frac{\pi}{\sin \frac{1}{4}\pi} = \frac{\pi}{\sqrt{2}},$$

using the notation and technique in Section 11A, pages 95–97 of the textbook. \square