Ma 404: Numerical Methods
Model Solutions to Midterm Examination

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Problem 1. Find the sum of the series

$$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

Answer. $1/(1 - \frac{2}{3}) = 3$.

Problem 2. Suppose $f(h) = 1 + 2h + 3h^2 + O(h^3)$, while $g(h) = 1 - h^2 + O(h^4)$ as $h \to 0$. Give the best possible formula for $f(h)g(h)$, and determine its order of approximation as $h \to 0$.
Answer. Multiplying it out gives $f(h)g(h) = 1 + 2h + 2h^2 + O(h^3)$.

Problem 3. Solve the equation $x = g(x)$ for $g(x) = 4 + \sqrt[4]{x}$ to 5 significant digits.
Answer. Iteration from $x_0 = 5$ yields the values

n	x_n
0	5.00000000
1	5.49534878
2	5.53108329
3	5.53356628
4	5.53373836
5	5.53375028
6	5.53375111
7	5.53375116
8	5.53375117
9	5.53375117

This has evidently converged to 5 significant digits after 5 iterations.

Problem 4. Find the Newton-Raphson iteration formula for $f(x) = x^3 + 12x - 5$, and use it to compute the (unique) solution to $f(x) = 0$ to 5 significant digits.
Answer. The iteration formula is $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - (x_n^3 + 12x_n - 5)/(3x_n^2 + 12)$. Starting from $x_0 = 1$, we get the following table of values:

n	x_n
0	1.00000000
1	0.46666667
2	0.41121649
3	0.41088595
4	0.41088594
5	0.41088594

This has evidently converged to 5 significant digits of the answer after 3 iterations.

Problem 5. Find the inverse matrix A^{-1} for

$$A = \begin{pmatrix} 3 & 2 \\ 8 & 5 \end{pmatrix}.$$

Answer. The formula for 2×2 matrices tells us the answer is

$$A^{-1} = \frac{1}{3 \cdot 5 - 8 \cdot 2} \begin{pmatrix} 5 & -2 \\ -8 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 8 & -3 \end{pmatrix}.$$

Problem 6. Let $\mathbf{x} = (2, -1, 2)$ and $\mathbf{y} = (3, 4, 0)$ be two vectors in $\mathbf{R}^3$. Compute $\|\mathbf{x}\|$, $\mathbf{x} \cdot \mathbf{y}$, and the angle between $\mathbf{x}$ and $\mathbf{y}$.

Answer. $\|\mathbf{x}\| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$. $\mathbf{x} \cdot \mathbf{y} = (2)(3) + (-1)(4) + (2)(0) = 2$.

Denote the angle between $\mathbf{x}$ and $\mathbf{y}$ by θ. Then

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} = \frac{2}{3\sqrt{3^2 + 4^2 + 5^2}} = \frac{2}{15} \implies \theta = 1.43706.$$

Problem 7. Find a factorization $A = LU$, where L is unit lower triangular and U is upper triangular, for

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 8 & -1 \\ 0 & 3 & 1 \end{pmatrix}.$$

Answer. This may be done by hand:

$$\begin{pmatrix} 1 & 5 & 3 \\ 2 & 8 & -1 \\ 0 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 3 \\ 2 & -2 & -7 \\ 0 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 3 \\ 2 & -2 & -7 \\ 0 & -3/2 & -19/2 \end{pmatrix}$$

$$\implies A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & 3 \\ 0 & -2 & -7 \\ 0 & 0 & -19/2 \end{pmatrix}.$$

Problem 8. Find the quadratic Lagrange polynomial that interpolates $(0, 1)$, $(1, 1)$, and $(3, -1)$. Evaluate it at $x = 2$.

Answer. Write $L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(-1)(-3)} = \frac{1}{3}x^2 - \frac{4}{3}x + 1$; $L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1)(-2)} = -\frac{1}{2}x^2 + \frac{3}{2}x$; $L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3)(2)} = \frac{1}{6}x^2 - \frac{1}{6}x$. Then $P_2(x) = 1L_{2,0}(x) + 1L_{2,1}(x) + (-1)L_{2,2}(x) = -\frac{1}{3}x^2 + \frac{1}{3}x + 1$.

Thus $P_2(2) = 1/3$. Checking, we see that $P_2(0) = 1$, $P_2(1) = 1$, and $P_2(3) = -1$, as required.

Problem 9. Find the least squares curve of the form $y(x) = Ce^{Ax}$ for the points $(1, 3)$, $(2, 2)$, and $(5, 1)$.

Answer. Substitute $Y = \log y, X = x, B = \log C$ to get a linear least squares problem for $Y = AX + B$ through the points $(1, \log 3)$, $(2, \log 2)$, and $(5, 0)$. This gives $A = -0.26459061$ and $B = 1.30282811$. Then $C = e^B = 3.67968853$, so the curve fitted to the original coordinates is

$$y = Ce^{Ax} = 3.67968853 * \exp(-0.26459061 * x),$$

which goes through the points $(1, 2.82423492), (2, 2.16765707), (5, 0.98007623)$.