You may use a calculator, the textbook, your class notes, and your homework sets. Please write your answers in the bluebook.

Problem 1. Express \( \sqrt{17} \) in binary using at least ten bits.

**Solution:** Compute \((2^{12} \sqrt{17})/2^{12} = (16888.24\ldots)/2^{12} = 100.0001111100\ldots\) (base 2).

Problem 2. Suppose \( f(h) = 1 - h^2 + O(h^6) \), while \( g(h) = 1 + h^2 + O(h^3) \) as \( h \to 0 \). Give the best possible formula for \( f(h) + g(h) \), and determine its order of approximation as \( h \to 0 \).

**Solution:** Compute \( f(h) + g(h) = 2 + O(h^6) + O(h^3) \), but \( O(h^3) + O(h^6) = O(h^3) \) as \( h \to 0 \), so the best possible formula is \( f(h) + g(h) = 2 + O(h^3) \).

Problem 3. (a) Solve the equation \( 3x - \cos x = 5 \) for \( x \) to 4 significant digits. (b) Prove that your method gives the required accuracy.

**Solution:** (b) Convert the equation to a fixed point problem for \( g \) defined by:

\[
3x - \cos x = 5 \quad \Rightarrow \quad x = \frac{\cos x + 5}{3} = g(x).
\]

Compute \( |g'(x)| = \left| -\frac{1}{3} \sin x \right| \leq \frac{1}{3} < 1 \) for all real \( x \). Hence the equation is solved by the fixed point of \( x = g(x) \), which is obtained by iteration. Since the contraction factor is \( 1/3 \) and the solution lies in the range \([1, 2]\) of \( g \), the solution will require about \( \log_3 \frac{|2 - 1|}{0.5 \times 10^{-4}} \approx 10 \) iterations.

(b) Iterate 20 times to be sure, from an initial value of 2. Answer is 1.642714576777085 \ldots .

Problem 4. Let

\[
A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]

(a) Find the determinant \( \det A \).

(b) Solve the linear system \( Ax = b \) for \( x \), given \( b = (1, 0, 0) \).

(c) Find the inverse matrix \( A^{-1} \).

**Solution:** (a) \( \det A = 2 \).

(b) Back substitution:

\[
x_3 = 0; \quad x_2 = -1/2; \quad x_1 = 1/2.
\]
(c) Solve $Ax = b$ for $b = (1, 0, 0), (0, 1, 0), (0, 0, 1)$ to get

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -2 & 4 \end{pmatrix}.$$ 

**Problem 5.** Let $x = (1, -1, 2)$ and $y = (3, 0, 4)$ be two vectors in $\mathbb{R}^3$. Compute $\|x\|$, $\|y\|$, $x \cdot y$, and the angle between $x$ and $y$.

**Solution:** $\|x\| = \sqrt{6}$, $\|y\| = 5$, $x \cdot y = 11$, and the angle between $x$ and $y$ is $\cos^{-1} \frac{11}{5\sqrt{6}} \approx 0.45526 \approx 26^\circ$.

**Problem 6.** Find a factorization $A = LU$, where $L$ is unit lower triangular and $U$ is upper triangular, for

$$A = \begin{pmatrix} 9 & 3 & 0 \\ 3 & 8 & -1 \\ 0 & 1 & 7 \end{pmatrix}.$$ 

(Hint: no interchanges are needed.)

**Solution:**

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{7} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 9 & 3 & 0 \\ 0 & 7 & -1 \\ 0 & 0 & \frac{50}{7} \end{pmatrix}.$$ 

**Problem 7.** Find the quadratic Lagrange polynomial that interpolates $(1, 5)$, $(2, 6)$, and $(3, -3)$. Write it in Newton’s form, and also as $ax^2 + bx + c$.

Evaluate it at $x = 1.5$.

**Solution:** $p(x) = 5 + (1)(x - 1) + (-5)(x - 2)(x - 1) = -5x^2 + 16x - 6$, so $p(1.5) = 27/4$.

**Problem 8.** Find the least squares curve of the form $y(x) = \frac{A}{x} + B$ for the points $(1, 3)$, $(2, 2)$, and $(5, 1)$.

**Solution:** Linearize by $X_k \leftarrow (1/x_k)$, $Y_k \leftarrow y_k$ and use the normal equations:

$$\left( \frac{\sum_k X_k^2}{\sum_k X_k} \quad \frac{\sum_k X_k}{N} \right) \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sum_k X_k Y_k \end{pmatrix}.$$ 

But $N = 3$, $\sum_k X_k = 1 + 0.5 + 0.2 = 1.7$, $\sum_k X_k^2 = 1 + 0.25 + 0.04 = 1.29$, $\sum_k Y_k = 3 + 2 + 1 = 6$, and $\sum_k X_k Y_k = 3 + 1 + 0.2 = 4.2$. Hence the normal equations are

$$\begin{pmatrix} 1.29 & 1.7 \\ 1.7 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 4.2 \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{0.98} \begin{pmatrix} 3 & -1.7 \\ -1.7 & 1.29 \end{pmatrix} \begin{pmatrix} 4.2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2.449 \\ 0.612 \end{pmatrix}.$$ 

Hence the least-squares curve is $y = \frac{2.449}{x} + 0.612$. 
