

Math 449: Numerical Applied Mathematics
Midterm Examination

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15 OCTOBER 2008

You may use a calculator, the textbook, your class notes, and your homework sets. Please write your answers in the bluebook.

Problem 1. Express $\sqrt{17}$ in binary using at least ten bits.

Solution: Compute $(2^{12}\sqrt{17})/2^{12} = (16888.24\dots)/2^{12} = 100.0001\ 1111\ 1000\dots$ (base 2).

Problem 2. Suppose $f(h) = 1 - h^2 + O(h^6)$, while $g(h) = 1 + h^2 + O(h^3)$ as $h \rightarrow 0$. Give the best possible formula for $f(h) + g(h)$, and determine its order of approximation as $h \rightarrow 0$.

Solution: Compute $f(h) + g(h) = 2 + O(h^6) + O(h^3)$, but $O(h^3) + O(h^6) = O(h^3)$ as $h \rightarrow 0$, so the best possible formula is $f(h) + g(h) = 2 + O(h^3)$.

Problem 3. (a) Solve the equation $3x - \cos x = 5$ for x to 4 significant digits. (b) Prove that your method gives the required accuracy.

Solution: (b) Convert the equation to a fixed point problem for g defined by:

$$3x - \cos x = 5 \implies x = \frac{\cos x + 5}{3} = g(x).$$

Compute $|g'(x)| = |-\frac{1}{3}\sin x| \leq \frac{1}{3} < 1$ for all real x . Hence the equation is solved by the fixed point of $x = g(x)$, which is obtained by iteration. Since the contraction factor is $1/3$ and the solution lies in the range $[1, 2]$ of g , the solution will require about

$$\log_3 \frac{|2 - 1|}{0.5 \times 10^{-4}} \approx 10$$

iterations.

(b) Iterate 20 times to be sure, from an initial value of 2. Answer is 1.6427 14576777085

Problem 4. Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(a) Find the determinant $\det A$.

(b) Solve the linear system $Ax = b$ for x , given $b = (1, 0, 0)$.

(c) Find the inverse matrix A^{-1} .

Solution: (a) $\det A = 2$.

(b) Back substitution:

$$x_3 = 0; \quad x_2 = -1/2; \quad x_1 = 1/2.$$

(c) Solve $Ax = b$ for $b = (1, 0, 0), (0, 1, 0), (0, 0, 1)$ to get

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -2 & 4 \end{pmatrix}.$$

Problem 5. Let $\mathbf{x} = (1, -1, 2)$ and $\mathbf{y} = (3, 0, 4)$ be two vectors in \mathbf{R}^3 . Compute $\|\mathbf{x}\|$, $\|\mathbf{y}\|$, $\mathbf{x} \cdot \mathbf{y}$, and the angle between \mathbf{x} and \mathbf{y} .

Solution: $\|\mathbf{x}\| = \sqrt{6}$, $\|\mathbf{y}\| = 5$, $\mathbf{x} \cdot \mathbf{y} = 11$, and the angle between \mathbf{x} and \mathbf{y} is $\cos^{-1} \frac{11}{5\sqrt{6}} \approx 0.45526 \approx 26^\circ$.

Problem 6. Find a factorization $A = LU$, where L is unit lower triangular and U is upper triangular, for

$$A = \begin{pmatrix} 9 & 3 & 0 \\ 3 & 8 & -1 \\ 0 & 1 & 7 \end{pmatrix}.$$

(Hint: no interchanges are needed.)

Solution:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{7} & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 9 & 3 & 0 \\ 0 & 7 & -1 \\ 0 & 0 & \frac{50}{7} \end{pmatrix}.$$

Problem 7. Find the quadratic Lagrange polynomial that interpolates $(1, 5)$, $(2, 6)$, and $(3, -3)$. Write it in Newton's form, and also as $ax^2 + bx + c$.

Evaluate it at $x = 1.5$.

Solution: $p(x) = 5 + (1)(x - 1) + (-5)(x - 2)(x - 1) = -5x^2 + 16x - 6$, so $p(1.5) = 27/4$.

Problem 8. Find the least squares curve of the form $y(x) = \frac{A}{x} + B$ for the points $(1, 3)$, $(2, 2)$, and $(5, 1)$.

Solution: Linearize by $X_k \leftarrow (1/x_k)$, $Y_k \leftarrow y_k$ and use the normal equations:

$$\begin{pmatrix} \sum_k X_k^2 & \sum_k X_k \\ \sum_k X_k & N \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sum_k X_k Y_k \\ \sum_k Y_k \end{pmatrix}.$$

But $N = 3$, $\sum_k X_k = 1 + 0.5 + 0.2 = 1.7$, $\sum_k X_k^2 = 1 + 0.25 + 0.04 = 1.29$, $\sum_k Y_k = 3 + 2 + 1 = 6$, and $\sum_k X_k Y_k = 3 + 1 + 0.2 = 4.2$. Hence the normal equations are

$$\begin{pmatrix} 1.29 & 1.7 \\ 1.7 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 4.2 \\ 6 \end{pmatrix} \implies \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{0.98} \begin{pmatrix} 3 & -1.7 \\ -1.7 & 1.29 \end{pmatrix} \begin{pmatrix} 4.2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2.449 \\ 0.612 \end{pmatrix}.$$

Hence the least-squares curve is $y = \frac{2.449}{x} + 0.612$.