

**Math 449: Numerical Applied Mathematics  
Midterm Examination**

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14 OCTOBER 2009

You may use a calculator, the textbook, your class notes, and the model homework solutions published this semester. Please write your answers in the bluebook.

**Problem 1.** Express 1.01011 01111 110 (base 2) in base 10 notation, giving at least four significant digits.

**Solution:** Compute the answer as

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \frac{1}{2048} + \frac{1}{4096} = 1.359130859375$$

[Note: This is approximately  $e/2 \approx 1.3591$ .]

**Problem 2.** (a) Find a polynomial  $p = p(h)$  of minimal degree in  $h$  such that  $\ln(1+h) = p(h) + O(h^3)$  as  $h \rightarrow 0$ . (b) Find  $\epsilon > 0$  such that  $|\ln(1+h) - p(h)| < 0.0005$  whenever  $|h| < \epsilon$ .

**Solution:** (a) Taylor's theorem gives the formula  $\ln(1+h) = h - \frac{1}{2}h^2 + \frac{1}{3(1+c)^3}h^3$  for some  $|c| < |h|$ , so  $\ln(1+h) = p(h) + O(h^3)$  as  $h \rightarrow 0$  where  $p(h) = h - h^2/2$ .

(b) Look only at  $|h| < \epsilon < 1$  to avoid trouble at  $1+h=0$ . But then  $|c| < |h| < \epsilon$  implies  $|1/(3(1+c)^3)| < 1/(3(1-\epsilon)^3)$ , so the error term is  $|\ln(1+h) - p(h)| < \epsilon^3/(3(1-\epsilon)^3)$  for  $|h| < \epsilon$ . Solve  $\epsilon^3/(3(1-\epsilon)^3) < 0.0005$  to get  $\epsilon < 0.0015^{1/3}/(1+0.0015^{1/3}) = 0.1027\dots$ . Evidently  $\epsilon = 0.1$  is sufficient to insure that the error is less than 0.0005.

**Problem 3.** (a) Solve the equation  $9x - \cos x = 4$  for  $x$  to four significant digits. (b) Prove that your method gives the required accuracy.

**Solution:** (a) Convert the equation to a fixed point problem for  $g$  defined by:

$$9x - \cos x = 4 \implies x = \frac{\cos x + 4}{9} = g(x).$$

Compute  $|g'(x)| = |-\frac{1}{9}\sin x| \leq \frac{1}{9} < 1$  for all real  $x$ . Hence the equation is solved by the fixed point of  $x = g(x)$ , which is obtained by iteration. The approximate root is  $x \approx 0.53976$

(b) Since the contraction factor is  $1/9$  and the solution lies in the range  $[0, 1]$  of  $g$ , the error after  $n$  iterations will be at most

$$\frac{|1 - 0|(\frac{1}{9})^n}{1 - \frac{1}{9}}.$$

It is easy to check that  $n = 5$  makes this expression smaller than  $2 \times 10^{-5} < 0.5 \times 10^{-4}$ , so five iterations suffice. Perform seven just to be sure, starting from  $x = 0$ , getting the approximate root  $x = 0.5397591546\dots$

**Problem 4.** Let  $\mathbf{x} = (1, -1, 2)$  and  $\mathbf{y}$  be two vectors in  $\mathbf{R}^3$ . Suppose  $\mathbf{y} \cdot \mathbf{y} = 6$  and  $\mathbf{x} \cdot (2\mathbf{y}) = 3$ . Compute the secant of the angle between  $\mathbf{x}$  and  $\mathbf{y}$ .

**Solution:** Let  $\theta$  be the angle between  $\mathbf{x}$  and  $\mathbf{y}$ . Then

$$\cos \theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} = \frac{\frac{1}{2}(3)}{(\sqrt{6})(\sqrt{6})} = \frac{1}{4}, \quad \implies \sec \theta = 4.$$

For the following three problems, let

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

**Problem 5.** Find a factorization  $A = LU$ , where  $L$  is unit lower triangular and  $U$  is upper triangular. (Hint: no interchanges are needed.)

**Solution:**

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix}.$$

**Problem 6.** Find the determinant  $\det A$ .

**Solution:** Multiply the diagonal elements of  $U$  to get  $\det(A) = 5$ .

**Problem 7.** Solve the linear system  $Ax = b$  for  $x$ , given  $b = (0, 0, 0, 0)$ .

**Solution:** Since  $A$  is invertible (its determinant is nonzero), the unique solution is  $x = (0, 0, 0, 0)$ .

**Problem 8.** Find the complex exponential Fourier series for the function  $f(x) = \cos^2(x)$ . (Hint: it has finitely many nonzero terms.)

**Solution:** One can do the integrals by elementary methods, but trigonometric identities provide an easier way:

$$\cos^2(x) = \frac{1}{2}[\cos(2x) + 1] = \frac{1}{2}\left[\frac{1}{2}(e^{i2x} + e^{-i2x}) + 1\right] = \frac{1}{4}e^{-i2x} + \frac{1}{2} + \frac{1}{4}e^{i2x}.$$

Hence the only nonzero coefficients in  $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$  are  $c_0 = \frac{1}{2}$  and  $c_{-2} = c_2 = \frac{1}{4}$ .

**Problem 9.** Find the quadratic Lagrange polynomial that interpolates  $(1, 1)$ ,  $(2, 2)$ , and  $(3, 5)$ . Write it in Newton's form and also as  $ax^2 + bx + c$ .

**Solution:** Form the divided difference table to obtain  $p(x) = 1 + (1)(x - 1) + (1)(x - 2)(x - 1) = x^2 - 2x + 2$ .

**Problem 10.** Find the least squares curve of the form  $y(x) = Ax + B$  for the points  $(1, 1)$ ,  $(2, 2)$ , and  $(3, 5)$ .

**Solution:** Use the normal equations:

$$\begin{pmatrix} \sum_k X_k^2 & \sum_k X_k \\ \sum_k X_k & N \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sum_k X_k Y_k \\ \sum_k Y_k \end{pmatrix}.$$

But  $N = 3$ ,  $\sum_k X_k = 1 + 2 + 3 = 6$ ,  $\sum_k X_k^2 = 1 + 4 + 9 = 14$ ,  $\sum_k Y_k = 1 + 2 + 5 = 8$ , and  $\sum_k X_k Y_k = 1 + 4 + 15 = 20$ . Hence the normal equations are

$$\begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 20 \\ 8 \end{pmatrix} \implies \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 20 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ -4/3 \end{pmatrix}.$$

Hence the least-squares curve is  $y = 2x - 4/3$ .