

**Math 449: Numerical Applied Mathematics**  
**Midterm Examination**

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You may use a calculator and the textbook. Please write your answers in the bluebook.

**Problem 1.** Express  $1/7 = 0.\overline{142857}$  (base 10) in base 2 notation, giving at least 10 digits after the radix point.

**Solution:** We multiply by  $2^{12}$  to get 12 > 10 digits after the radix point:

$$(2^{12})(1/7) = 585.\overline{142857}$$

This rounds to the integer 585, which has the binary expansion  $585 = 1001001001$  (base 2). Shifting the radix point 12 places to the left yields  $x = 0.0010010010$  (base 2).

**Problem 2.** Note: “log” means the natural logarithm.

(a) Find a polynomial  $p = p(h)$  of minimal degree in  $h$  such that  $\log(1+h) = p(h) + O(h^5)$  as  $h \rightarrow 0$ .

(b) Find  $\epsilon > 0$  such that  $|\log(1+h) - p(h)| < 0.00007$  whenever  $|h| < \epsilon$ .

**Solution:** (a) Taylor’s theorem gives the formula  $\log(1+h) = h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4 + \frac{1}{5}h^5 \frac{1}{(1+c)^5}$  for some  $|c| < |h|$ , so  $\log(1+h) = p(h) + O(h^5)$  as  $h \rightarrow 0$  where  $p(h) = h - h^2/2 + h^3/3 - h^4/4$ .

(b) Look only at  $|h| < \epsilon < 0.5$  to get the bound  $|\frac{1}{(1+c)^5}| < 32$  since  $|c| < |h| < \epsilon$ . Then the error term is  $|\log(1+h) - p(h)| < 32\epsilon^5/5$  for all  $|h| < \epsilon$ . Solve  $\epsilon^5 < (0.00007)(5/32) \approx 0.00001$  to see that  $\epsilon < 0.1$  is sufficient to insure that the error is less than 0.00007.

**Problem 3.** The function  $f(x) = e^x + \log x$  has a unique root in the interval  $0 < x < 1$ .

(a) Find the Newton–Raphson iteration formula for the equation  $f(x) = 0$ .

(b) Solve for  $x$  in  $f(x) = 0$  by any method that you choose. Give at least 5 correct digits after the decimal point.

**Solution:** (a) Find  $f'(x) = e^x + 1/x$ . Then the Newton–Raphson iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\exp(x_n) + \log(x_n)}{\exp(x_n) + 1/x_n}.$$

(b) One fast method is iteration. Since the root  $x$  of  $e^x + \log x = 0$  solves  $\log x = -e^x$ , exponentiate both sides to get

$$x = 1/\exp(\exp(x)),$$

which requires only three calculator button-presses per iteration. Put  $x_0 = 0.5$  to get the following sequence of approximations:

0.5000000000000000  
 0.192295645547965  
 0.297592911581365  
 0.260119669765427  
 0.273327300419376  
 0.268654165852876  
 0.270305455800659  
 0.269721684952626  
 0.269928027955766  
 0.269855088504433  
 0.269880871077923  
 0.269871757409811  
 0.269874978916463  
 0.269873840174690  
 0.269874242698275  
 0.269874100413818  
 0.269874150708674  
 0.269874132930396  
 0.269874139214680  
 0.269874136993305  
 0.269874137778519  
 0.269874137500961  
 0.269874137599072  
 0.269874137564392  
 0.269874137576651  
 0.269874137572317

Evidently, the first five digits of  $x$  after the decimal point are 0.26987.

For the following four problems, let  $A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 4 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ , and  $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ .

**Problem 4.** Find a factorization  $A = LU$ , where matrix  $L$  is unit lower triangular and matrix  $U$  is upper triangular. (Hint: no interchanges are needed.)

**Solution:**

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix}.$$

**Problem 5.** Find the determinant  $\det A$ .

**Solution:** Multiply the diagonal elements of  $U$  to get  $\det(A) = 12$ .

**Problem 6.** Let  $L$  and  $U$  be the matrices from problem 4. Suppose that  $\mathbf{y}$  and  $\mathbf{z}$  are vectors in  $\mathbf{R}^3$ ,  $\mathbf{y}$  solves  $L\mathbf{y} = 2\mathbf{x}$ , and  $\mathbf{z}$  solves  $U\mathbf{z} = 3\mathbf{y}$ . Compute  $A\mathbf{z}$ .

**Solution:** Since  $A = LU$ , compute  $A\mathbf{z} = L(U\mathbf{z}) = L(3\mathbf{y}) = 6\mathbf{x}$ .

**Problem 7.** (a) Compute the vector  $A\mathbf{b}$ .

(b) Compute the inner product of  $A\mathbf{b}$  with  $\mathbf{b}$ .

(c) Find the cosine of the angle between  $A\mathbf{b}$  and  $\mathbf{b}$

**Solution:**

(a)  $A\mathbf{b} = (6, -11, 10)$ .

(b)  $(A\mathbf{b}) \cdot \mathbf{b} = 48$ .

(c) The cosine of the angle is

$$\frac{(A\mathbf{b}) \cdot \mathbf{b}}{\|A\mathbf{b}\| \|\mathbf{b}\|} = \frac{48}{3\sqrt{257}} = \frac{16}{\sqrt{257}} \approx 0.99805,$$

since  $\|A\mathbf{b}\| = \sqrt{257}$  and  $\|\mathbf{b}\| = 3$ .

**Problem 8.** Find the complex exponential Fourier series for the function  $f(x) = \cos(x) + \sin(2x)$ . (Hint: it has finitely many nonzero terms.)

**Solution:** One can do the integrals by elementary methods, but Euler's formula  $e^{ix} = \cos(x) + i\sin(x)$  and the identity  $1/i = -i$  provide an easier way:

$$\cos(x) + \sin(2x) = \frac{e^{ix} + e^{-ix}}{2} + \frac{e^{2ix} - e^{-2ix}}{2i} = \frac{i}{2}e^{-2ix} + \frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix} - \frac{i}{2}e^{2ix}.$$

Hence the only nonzero coefficients in  $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$  are  $c_{-2} = \frac{i}{2}$ ,  $c_{-1} = \frac{1}{2}$ ,  $c_1 = \frac{1}{2}$ , and  $c_2 = -\frac{i}{2}$ .

**Problem 9.** Fix  $c > 0$  and let  $P_0 = (-1, 0)$ ,  $P_1 = (0, c)$ , and  $P_2 = (1, 0)$  be three points in the  $(x, y)$ -plane. Let  $B = B(t)$ ,  $0 \leq t \leq 1$ , be the Bézier curve with control points  $P_0, P_1, P_2$ . Find the maximum  $y$ -coordinate of  $B(t)$  for any  $t$ .

**Solution:** From the definition,

$$\begin{aligned} B(t) &= \binom{2}{0} t^2 P_0 + \binom{2}{1} t(1-t) P_1 + \binom{2}{2} (1-t)^2 P_2 = t^2(-1, 0) + 2t(1-t)(0, c) + (1-t)^2(1, 0) \\ &= (-t^2 + (1-t)^2, 2t(1-t)c) = (x(t), y(t)). \end{aligned}$$

The  $y$ , or second, coordinate  $y(t) = 2t(1-t)c$  has its maximum for  $0 \leq t \leq 1$  at the point  $t = 1/2$ , with maximum value  $y(1/2) = c/2$ .

**Problem 10.** Find the least squares curve of the form  $y(x) = Ax + B$  for the three points  $(0, 0)$ ,  $(0, 1)$ , and  $(2, 1)$ .

**Solution:** Use the normal equations:

$$\begin{pmatrix} \sum_k X_k^2 & \sum_k X_k \\ \sum_k X_k & N \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sum_k X_k Y_k \\ \sum_k Y_k \end{pmatrix}.$$

But  $N = 3$ ,  $\sum_k X_k = 0 + 0 + 2 = 2$ ,  $\sum_k X_k^2 = 0 + 0 + 4 = 4$ ,  $\sum_k Y_k = 0 + 1 + 1 = 2$ , and  $\sum_k X_k Y_k = 0 + 0 + 2 = 2$ . Hence the normal equations are

$$\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \implies \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix}.$$

Hence the least-squares curve is  $y = \frac{1}{4}x + \frac{1}{2}$ .