

Ma 450: Mathematics for Multimedia

Homework Assignment 5

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Due Friday, April 13th, 2012

1. Draw the graphs of $w(\frac{t}{3} - 4)$ and $w(\frac{t-4}{3})$ on one set of axes for the Haar function $w(t)$ defined in Equation 5.2.
2. For each $(a, b) \in \mathbf{Aff}$, define the linear operator $\zeta(a, b) : L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$ by

$$\zeta(a, b)f(t) \stackrel{\text{def}}{=} \sqrt{a} f(at + b).$$

Is ζ a representation of the group \mathbf{Aff} ? If so, is it a faithful representation? Is it unitary?

3. Let $f = f(\mathbf{a}) = f(a, b)$ be the function on \mathbf{Aff} defined by $f(\mathbf{a}) = \mathbf{1}_D(\mathbf{a})$, where $\mathbf{1}_D$ is the indicator function of the region $D = \{\mathbf{a} = (a, b) : A < a < A', B < b < B'\} \subset \mathbf{Aff}$ for $0 < A < A'$ and $-\infty < B < B' < \infty$. Evaluate $\int_{\mathbf{Aff}} f(\mathbf{a}) d\mathbf{a}$ using the normalized left-invariant integral on \mathbf{Aff} .
4. Let $w = w(t)$ be the Haar mother function and define

$$\phi_{M,K}^J(t) \stackrel{\text{def}}{=} \sum_{j=M+1}^{M+J} \frac{1}{2^j} w\left(\frac{t-K}{2^j}\right)$$

for arbitrary fixed $K \in \mathbf{R}$ and $M, J \in \mathbf{Z}$ with $J > 0$. Prove that

$$\lim_{J \rightarrow \infty} \phi_{M,K}^J(t) = 2^{-M} \mathbf{1}_{[K, K+2^M)}(t) \stackrel{\text{def}}{=} \phi_{M,K}(t),$$

and also that $\langle \phi_{M,K}^J, u \rangle \rightarrow \langle \phi_{M,K}, u \rangle$ as $J \rightarrow \infty$ for any function $u \in L^2(\mathbf{R})$.

5. Let d be a positive integer. Show that the following function has $d - 1$ continuous derivatives:

$$\phi(\xi) = \begin{cases} \xi^d e^{-\xi^2}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi \leq 0. \end{cases}$$

6. Fix an integer $d > 0$ and define

$$\phi(\xi) = \begin{cases} \exp [-(\log \xi)^{2d}], & \text{if } \xi > 0; \\ 0, & \text{if } \xi \leq 0. \end{cases}$$

Show that for any $n, m \in \mathbf{N}$, we have $\phi^{(n)}(\xi) = O(1/|\xi|^m)$ as $\xi \rightarrow \pm\infty$.

7. Compute $\|w\|$, where

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log \xi)^2}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi \leq 0. \end{cases}$$

(Hint: use Plancherel's theorem and Equation B.6 in Appendix B.)

8. Let w be the function defined by

$$\mathcal{F}w(\xi) = \begin{cases} e^{-(\log |\xi|)^2}, & \text{if } \xi \neq 0; \\ 0, & \text{if } \xi = 0. \end{cases}$$

Show that w is admissible and compute its normalization constant c_w .

9. Fix $A < 0$, $B > 0$, and $R > 1$ and suppose that $w = w(x)$ is a function satisfying $\mathcal{F}w(\xi) = 1$ if $RA < \xi < A$ or $B < \xi < RB$, with $\mathcal{F}w(\xi) = 0$, otherwise. Show that w satisfies the admissibility condition of Theorem 5.2, and compute the normalization constant c_w . Give a formula for w .

10. Show that if $h = \{h(k) : k \in \mathbf{Z}\}$ is a self-orthonormal filter, and M is any fixed integer, then the sequence defined by

$$g(k) = (-1)^k \overline{h(2M - 1 - k)}, \quad \text{for all } k \in \mathbf{Z},$$

satisfies the completeness condition of Equation 5.45.

11. a. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the antisymmetry condition $h(0) = -h(3)$ and $h(1) = -h(2)$?

b. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the symmetry condition $h(0) = h(3)$ and $h(1) = h(2)$?

12. Suppose that an orthogonal MRA has a scaling function ϕ satisfying $\phi(t) = 0$ for $t \notin [a, b]$. Prove that the low-pass filter h for this MRA must satisfy $h(n) = 0$ for all $n \notin [2a - b, 2b - a]$. (This makes explicit the finite support of h in Equation 5.36.)

13. Suppose that $h = \{h(k) : k \in \mathbf{Z}\}$ and $g = \{g(k) : k \in \mathbf{Z}\}$ satisfy the orthogonal CQF conditions. Show that the 2-periodizations h_2, g_2 of h and g are the Haar filters. Namely, show that $h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2}$.

14. Let ϕ be the scaling function of an orthogonal MRA, and let ψ be the associated mother function. For $(x, y) \in \mathbf{R}^2$, define

$$\begin{aligned} e_0(x, y) &= \phi(x)\phi(y), & e_1(x, y) &= \phi(x)\psi(y) \\ e_2(x, y) &= \psi(x)\phi(y), & e_3(x, y) &= \psi(x)\psi(y). \end{aligned}$$

Prove that the functions $\{e_n : n = 0, 1, 2, 3\}$ are orthonormal in $L^2(\mathbf{R}^2)$, the inner product space of square-integrable functions on \mathbf{R}^2 .