1. Draw the graphs of \( w\left(\frac{t}{3} - 4\right) \) and \( w\left(\frac{t-4}{3}\right) \) on one set of axes for the Haar function \( w(t) \) defined in Equation 5.2.

2. For each \((a,b) \in \text{Aff}\), define the linear operator \( \zeta(a,b) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) by

\[
\zeta(a,b) f(t) \overset{\text{def}}{=} \sqrt{a} f(at + b).
\]

(a) Is \( \zeta(a,b) \) unitary for each \((a,b) \in \text{Aff}\)?

(b) Is \((a,b) \mapsto \zeta(a,b)\) a one-to-one mapping on \( \text{Aff}\)?

(c) Is \( \zeta \) a representation of the group \( \text{Aff}\)?

3. Let \( f = f(a) = f(a,b) \) be the function on \( \text{Aff} \) defined by \( f(a) = 1_D(a) \), where \( 1_D \) is the indicator function of the region \( D = \{a = (a,b) : A < a < A', B < b < B'\} \subset \text{Aff} \) for \( 0 < A < A' \) and \( -\infty < B < B' < \infty \). Evaluate \( \int_{\text{Aff}} f(a) \, da \) using the normalized left-invariant integral on \( \text{Aff}\).

4. Let \( w = w(t) \) be the Haar mother function and define

\[
\phi_{M,K}^J(t) \overset{\text{def}}{=} \sum_{j=M+1}^{M+J} \frac{1}{2^j} w\left(\frac{t-K}{2^j}\right)
\]

for arbitrary fixed \( K \in \mathbb{R} \) and \( M,J \in \mathbb{Z} \) with \( J > 0 \). Prove that

\[
\lim_{J \to \infty} \phi_{M,K}^J(t) = 2^{-M} 1_{[K,K+2^M]}(t) \overset{\text{def}}{=} \phi_{M,K}(t),
\]

and also that \( \langle \phi_{M,K}^J, u \rangle \to \langle \phi_{M,K}, u \rangle \) as \( J \to \infty \) for any function \( u \in L^2(\mathbb{R}) \).
5. Let $d$ be a positive integer. Show that the following function has $d - 1$ continuous derivatives:

$$
\phi(\xi) = \begin{cases} 
\xi^d e^{-\xi^2}, & \text{if } \xi > 0; \\
0, & \text{if } \xi \leq 0.
\end{cases}
$$

6. Fix an integer $d > 0$ and define

$$
\phi(\xi) = \begin{cases} 
\exp \left[ - (\log \xi)^2 \right], & \text{if } \xi > 0; \\
0, & \text{if } \xi \leq 0.
\end{cases}
$$

Show that for any $n, m \in \mathbb{N}$, we have $\phi^{(n)}(\xi) = O(1/|\xi|^m)$ as $\xi \to \pm \infty$.

7. Compute $\|w\|$, where

$$
\mathcal{F}w(\xi) = \begin{cases} 
e^{-\left(\log |\xi|\right)^2}, & \text{if } \xi \neq 0; \\
0, & \text{if } \xi = 0.
\end{cases}
$$

(Hint: use Plancherel’s theorem and Equation B.6 in Appendix B.)

8. Let $w$ be the function defined by

$$
\mathcal{F}w(\xi) = \begin{cases} 
e^{-(\log |\xi|)^2}, & \text{if } \xi \neq 0; \\
0, & \text{if } \xi = 0.
\end{cases}
$$

Show that $w$ is admissible and compute its normalization constant $c_w$.

9. Fix $A < 0$, $B > 0$, and $R > 1$ and suppose that $w = w(x)$ is a function satisfying $\mathcal{F}w(\xi) = 1$ if $RA < \xi < A$ or $B < \xi < RB$, with $\mathcal{F}w(\xi) = 0$, otherwise. Show that $w$ satisfies the admissibility condition of Theorem 5.2, and compute the normalization constant $c_w$. Give a formula for $w$.

10. Show that if $h = \{h(k) : k \in \mathbb{Z}\}$ is a self-orthonormal filter, and $M$ is any fixed integer, then the sequence defined by

$$
g(k) = (-1)^k h(2M - 1 - k), \quad \text{for all } k \in \mathbb{Z},
$$

satisfies the completeness condition of Equation 5.45.

11. a. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the antisymmetry condition $h(0) = -h(3)$ and $h(1) = -h(2)$?

b. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the symmetry condition $h(0) = h(3)$ and $h(1) = h(2)$?
12. Suppose that an orthogonal MRA has a scaling function $\phi$ satisfying $\phi(t) = 0$ for $t \notin [a, b]$. Prove that the low-pass filter $h$ for this MRA must satisfy $h(n) = 0$ for all $n \notin [2a - b, 2b - a]$. (This makes explicit the finite support of $h$ in Equation 5.36.)

13. Suppose that $h = \{h(k) : k \in \mathbb{Z}\}$ and $g = \{g(k) : k \in \mathbb{Z}\}$ satisfy the orthogonal CQF conditions. Show that the 2-periodizations $h_2, g_2$ of $h$ and $g$ are the Haar filters. Namely, show that $h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2}$.

14. Let $\phi$ be the scaling function of an orthogonal MRA, and let $\psi$ be the associated mother function. For $(x, y) \in \mathbb{R}^2$, define

$$
e_0(x, y) = \phi(x)\phi(y), \quad e_1(x, y) = \phi(x)\psi(y)$$
$$e_2(x, y) = \psi(x)\phi(y), \quad e_3(x, y) = \psi(x)\psi(y).$$

Prove that the functions $\{e_n : n = 0, 1, 2, 3\}$ are orthonormal in $L^2(\mathbb{R}^2)$, the inner product space of square-integrable functions on $\mathbb{R}^2$. 