1. Fix an integer \( q > 0 \), let \( N = 2^q > 1 \) and consider a graph with vertices labeled \( 0, 1, \ldots, N-1 \). Suppose that vertex \( i \) is connected by an edge to vertex \( j \) if and only if the base-two expansions for \( i \) and \( j \) differ by exactly one bitflip. Compute the total number of edges.

2. Construct a prefix code for the alphabet \( A = \{a, b, c, d, e, f\} \) with codeword lengths 1,2,2,3,3,3 or prove that none exists.

3. Construct a prefix code for the 24-letter Greek alphabet \( A = \{\alpha, \beta, \gamma, \ldots, \omega\} \) with longest codeword 5, or prove that none exists.

4. Suppose we have two prefix codes, \( c_0(a,b) = (1,0) \) and \( c_1(a,b) = (0,1) \), for the alphabet \( A = \{a,b\} \). Show that the following dynamic encoding is uniquely decipherable by finding a decoding algorithm:

   **Simple Dynamic Encoding Example**

   ```
   dynamicencoding0( msg[], M ):
   [0] Initialize n=0
   [2] Transmit msg[m] using code n
   [3] If msg[m]=='b', then toggle n = 1-n
   ```

   (This encoding is called dynamic because the codeword for a letter might change as a message is encoded, in contrast with the static encodings studied in this chapter. It gives an example of a uniquely decipherable and instantaneous code which is nevertheless not a prefix code.)

5. Suppose that \( A \) is a finite set, \( s \) is a fixed positive integer, and \( p_k : A \rightarrow [0,1] \) is a probability function on \( A \) for each \( k = 1, \ldots, s \).

   (a) Show that the function \( p : A^s \rightarrow [0,1] \) defined by

   \[
   p(x_1, x_2, \ldots, x_s) \overset{\text{def}}{=} p_1(x_1)p_2(x_2)\cdots p_s(x_s)
   \]

   is a probability function on \( A^s \).

   (b) Compute the entropy \( H(p) \) in terms of \( H(p_1), \ldots, H(p_s) \).
6. A k-ary tree is called extended if every interior, or non-leaf, vertex has all \( k \) children. Count, with proof, the extended \( k \)-ary trees of depth 3 or less.

7. Fix a positive integer \( n \) and consider the alphabet \( A = \{ a_1, \ldots, a_n, a_{n+1} \} \) with occurrence probabilities \( p(a_i) = 2^{-i} \) for \( i = 1, \ldots, n \), and \( p(a_{n+1}) = 2^{-n} \). (a) Construct a Huffman code for the alphabet and compare its bit rate with \( H(p) \).
(b) Construct a canonical Huffman code for this alphabet, with the property that no letter has a codeword consisting of just 1-bits. Compute its bit rate.

8. What is the probability of an undetected error in 8 data bits in a \( 2 \times 2 \times 2 \) array with crossed parity checks if the data bits each have an independent probability \( p \) of being flipped, but the 12 top, front, and left face parity bits are known to be correct?

9. Find a binary code with two 10-bit or shorter codewords, wherein restoration to the nearest codeword corrects any three or fewer bit flips.

10. Prove that casting out seventeens will detect all one-digit errors in hexadecimal arithmetic. Find an example one-hexadecimal-digit error undetected by casting out fifteens.

11. Will the combination of checksums \( c_9 \) and \( c_{11} \) distinguish all nonequal 2-decimal-digit positive integers?

12. Find a mod-2 polynomial of degree 3 that is relatively prime to \( p(t) = t^7 + t^5 + t^3 + t \). (Hint: use Euclid’s algorithm for mod-2 polynomials.)

13. Suppose that \( b \) is a prime number. Write \( b = \ldots b_2 b_1 b_0 \) (base 2) and let \( p(t) = b_0 + b_1 t + b_2 t^2 + \cdots \) be the associated mod-2 polynomial. Prove or find a counterexample to the claim that \( p \) must be irreducible.

14. Suppose that \( s > 0 \) and \( a > 1 \) are integers with \( \gcd(a, s) = 1 \). Prove that there is some integer \( N > 0 \) such that \( a^N - 1 \) is divisible by \( s \), but \( s \) does not divide \( a^k - 1 \) for any positive integer \( k < N \). (This is Theorem 6.19 for integers rather than mod-2 polynomials.)

15. Find integers \( j, k, 0 < k < j < 32 \), such that \( s(t) = t^{32} + t^j + t^k + 1 \) is an irreducible mod-2 polynomial, or prove that none exists. (Hint: try dividing one such \( s(t) \) by \( t + 1 \).)