Ma 450: Mathematics for Multimedia

Solution: to Homework Assignment 4

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Due Friday, March 31st, 2017

1. Fix $x$ and find a formula for the value $y = f(x)$ of the Lagrange polynomial $f$ through the points $(-1,a)$, $(0,0)$, and $(1,b)$, in terms of $x$, $a$, and $b$. Then find $d^2y/dx^2$.

**Solution:** The Lagrange polynomial is

$$\Lambda_2(x) = - \frac{ax}{2} + \frac{bx}{2} + \frac{ax^2}{2} + \frac{bx^2}{2}$$

Its second derivative with respect to $x$ is $\frac{d^2}{dx^2}\Lambda_2(x) = a + b$.

2. Find the expansion in Chebyshev polynomials $T_0(x), T_1(x), T_2(x)$ of the function $f(x) = 1 - x^2$ defined for $x \in [-1,1]$.

**Solution:** Since $f$ itself is a polynomial of degree 2, it equals its Chebyshev polynomial expansion: $f(x) = c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x)$, for all $x \in [-1,1]$. The expansion coefficients may thus be found by the method of undetermined coefficients. But $T_0(x) = 1$, $T_1(x) = x$, and $T_2(x) = 2x^2 - 1$, so:

$$1 - x^2 = c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot (2x^2 - 1) = [c_0 - c_2] + [c_1]x + 2x^2[c_2],$$

so $c_1 = 0$, $c_2 = -\frac{1}{2}$, and $c_0 = \frac{1}{2}$.

3. Suppose that $x[]$ and $y[]$ are both increasing arrays of real numbers indexed by $0, 1, \ldots, N$. Implement $\text{pwlinear}(x[], y[], y, N)$, a function that returns the value $x$ satisfying $f(x) = y$, where $f$ is the piecewise linear function interpolating the set $\{(x_k, y_k) : k = 0, 1, \ldots, N\} \subset \mathbb{R}^2$.

**Solution:** Note that this is virtually the same as $\text{pwlinear}()$ with $x$ and $y$ interchanged.

**Piecewise Linear Inversion**

```python
pwlinear( x[], y[], y, N ):
[0] If y < y[0], then return x[0]
[1] For k=1 to N, do [2]
[2] If y < y[k], then return
   (x[k]*(y-y[k-1]) + x[k-1]*(y[k]-y)) / (y[k]-y[k-1])
[3] Return x[N]
```

This can be made more robust by checking that the arrays $x[]$ and $y[]$ are increasing.
4. Suppose $x_1 < x_2$ and fix $y_1 < 0$ and $y_2 > 0$. Let $f$ be the linear function interpolating the set \( \{(x_1, y_1), (x_2, y_2)\} \). On what interval (if any) is $f > 0$? On what interval (if any) is $f < 0$?

\textbf{Solution:} First use the point-slope formula through the points \((x_1, y_1), (x_2, y_2)\) to find $f(x) = m(x - x_1) + y_1$ by computing $m = (y_2 - y_1)/(x_2 - x_1)$. Note that $m > 0$ since both $y_2 - y_1 > 0$ and $x_2 - x_1 > 0$. Thus $f$ is strictly increasing.

Use Equation 4.14 to find $x_0 = \frac{2y_1 - y_2 - y_1}{y_1 - y_2}$, the root of the linear function $f = f(x)$. Since $f$ is strictly increasing, it must satisfy $f(t) < 0$ for $t \in (-\infty, x_0)$ and $f(t) > 0$ for $t \in (x_0, \infty)$. $\blacksquare$

5. Find the quantization error, imprecision, and inaccuracy of a counter that wraps pennies into bundles of 50, discarding leftovers, and reports the number wrapped as the total number of pennies. Is this counter a calibrated instrument?

\textbf{Solution:} The quantization error is half the 50-penny difference between possible reported values, or 25 pennies.

Since the reported count is never more than the number of pennies but may be up to 49 pennies too low, the expected ideal value is 24.5 pennies more than the reported value. Thus the counter is not calibrated.

The imprecision is the root mean square error of a uniform density on the interval \([0, 49]\) with mean value 24.5:

$$\sqrt{\frac{1}{50} \sum_{p=0}^{49} (p - 24.5)^2} = \sqrt{\frac{2}{50} \sum_{p=0}^{24} (p - 24.5)^2} = \sqrt{\frac{2}{50} \sum_{k=1}^{25} (2k - 1)^2},$$

after the change of variable \((p - 24.5)^2 \leftarrow [(2k - 1)/2]^2\). But the sum of the squares of the first $N$ odd integers is $\frac{1}{3} N(4N^2 - 1)$ by Equation 0.122(2) on page 2 in Gradshteyn and Ryzhik’s \textit{Table of Integrals, Series, and Products}, fifth edition (1994), Academic Press, San Diego; ISBN 0-12-294755-X. This is 20825 for $N = 25$. Hence the imprecision is $\sqrt{20825} \approx 14.43$ pennies.

The inaccuracy is the root mean square error between a uniform random variable taking integer values in the interval \([0, 49]\), and the measured value 0:

$$\sqrt{\frac{1}{50} \sum_{p=0}^{49} (p - 0)^2} = \sqrt{\frac{1}{50} \sum_{p=1}^{49} p^2}.$$

But the sum of the squares of the first $N$ integers is $\frac{1}{3} N(2N+1)(N+1)$ by Equation 0.121(2) on page 2 in Gradshteyn and Ryzhik’s \textit{Table of Integrals, Series, and Products}, fifth edition (1994), Academic Press, San Diego; ISBN 0-12-294755-X. This is 40425 for $N = 49$. Hence the inaccuracy is $\sqrt{40425} \approx 28.43$ pennies. $\blacksquare$

6. Fix $N \in \mathbb{Z}^+$ and let $s \in L^2([0, 1])$ be the piecewise constant function

$$s(t) \overset{\text{def}}{=} \sum_{k=0}^{2N-1} (-1)^k \mathbf{1}(2Nt - k), \quad 0 \leq t \leq 1,$$

where $\mathbf{1}$ is the indicator function of the interval \([0, 1]\). Let $f \in L^2([0, 1])$ be the function $f(t) \overset{\text{def}}{=} \sin(2\pi Nt)$ for $0 \leq t \leq 1$.

a. Compute $\|f\|^2$, $\|s\|^2$, and $\|f - s\|^2$. 

b. Treating $s$ as the signal and $f - s$ as the noise, compute the signal to noise ratio on $[0, 1]$.

c. Treating $f$ as the signal and $f - s$ as the noise, compute the signal to noise ratio on $[0, 1]$.

**Solution:**

a. First note that $\mathbf{1}(2Nt - k) = 1$ if and only if $t \in \left[\frac{k}{2N}, \frac{k+1}{2N}\right)$. On the interior $\left(\frac{k}{2N}, \frac{k+1}{2N}\right)$ of that interval, $\sin(2\pi Nt) > 0$ if $k$ is even, while $\sin(2\pi Nt) < 0$ if $k$ is odd. Hence $s(t) = \text{sgn}[f(t)]$ except at the zeroes $\frac{k}{2N} : 0 \leq k \leq 2N$ of $f$. Note that $|s(t)| = 1$ for all $t \in [0, 1)$.

Second, compute two of the squared norms by the Calculus:

$$\|f\|^2 = \int_0^1 |\sin(2\pi Nt)|^2 dt = \frac{1}{2}; \quad \|s\|^2 = \int_0^1 |s(t)|^2 dt = 1.$$

Third, note that $\|f - s\|^2 = \langle f - s, f - s \rangle = \|f\|^2 + \|s\|^2 - 2 \langle f, s \rangle$, since all functions are real-valued. But $f(t)s(t) = f(t)\text{sgn}[f(t)] = |f(t)|$ at all but $2N + 1$ points $t$, so

$$\langle f, s \rangle = \int_0^1 |f(t)| dt = \int_0^1 |\sin(2\pi Nt)| dt = \frac{2}{\pi}, \quad \Rightarrow \|f - s\|^2 = \frac{3}{\pi} - \frac{4}{\pi} \approx 0.22676,$$

again evaluated by the Calculus.

b. Use Equation 4.36 with $f_s = s$ and $f_q = f - s$ to compute $\text{SNR}(f) = 10\log_{10} \frac{1}{\frac{3}{\pi}} \approx 6.444$.

c. Use Equation 4.36 with $f_s = f$ and $f_q = f - s$ to compute $\text{SNR}(f) = 10\log_{10} \frac{1/2}{\frac{3}{\pi}} \approx 3.434$.

Note that none of the norms depend on $N$. 

7. Let $f = f(x, y)$ be the joint probability density supported on the region $R = \{(x, y) : 0 \leq y \leq 1, y - 1 \leq x \leq y + 1\}$ and defined by the formula $f(x, y) = 1 - |x - y|$ for $(x, y) \in R$, with $f(x, y) = 0$ elsewhere.

a. Show that $\int \int_R f(x, y) \, dx \, dy = 1$.

b. Compute the normalizing constant $c_y$ and determine $f(x \mid y)$.

c. Compute the expectation $E(x \mid y)$. Is $d(x) = x$ an unbiased estimator?

d. Compute the risk $R(d, y)$ for the decision function $d(x) = x$. Does it depend on $y$?

**Solution:**

a. The integral may be computed by iterated integration in each variable:

$$\int \int_R f(x, y) \, dx \, dy = \int_0^1 \left( \int_{y-1}^{y+1} (1 - |x - y|) \, dx \right) \, dy$$

$$= \int_0^1 \left( \int_{y-1}^y (1 + x - y) \, dx + \int_y^{y+1} (1 - x + y) \, dx \right) \, dy$$

$$= \int_0^1 \left( \left[ x + \frac{x^2}{2} - yx \right]_{y-1}^{y} + \left[ x - \frac{x^2}{2} + yx \right]_{y}^{y+1} \right) \, dy$$

$$= \int_0^1 \left( \left[ 1 - \frac{y^2}{2} - (y - 1)^2 \right]_{y-1}^{y} + \left[ 1 - \frac{(y + 1)^2}{2} - y^2 + y \right]_{y}^{y+1} \right) \, dy$$

$$= \int_0^1 1 \, dy = 1.$$

b. Using Equation 4.38, compute

$$c_y = \int_{-\infty}^\infty f(x, y) \, dx = \int_{y-1}^{y+1} (1 - |x - y|) \, dx = 1,$$
since the integrand is a hat function whose graph bounds a triangle of base 2 and height 1. Thus

\[ f(x \mid y) = f(x, y) = \begin{cases} 
1 - |x - y|, & \text{if } 0 \leq y \leq 1 \text{ and } y - 1 \leq x \leq y + 1; \\
0, & \text{otherwise.} 
\end{cases} \]

c. Using the result from part b, compute the expectation

\[
E(x \mid y) = \int_0^1 xf(x \mid y) \, dx = \int_{y-1}^{y+1} x(1 - |x - y|) \, dx = \int_{-1}^{+1} [x + y](1 - |x|) \, dx
\]

\[
= \int_{-1}^{+1} x(1 - |x|) \, dx + y \int_{-1}^{+1} (1 - |x|) \, dx = 0 + 1y = y.
\]

The third step follows from the substitution \( x \rightarrow x + y \). The integral of \( x(1 - |x|) \) on \([-1, 1]\) vanishes by antisymmetry, and the integral of the hat function \( 1 - |x| \) on \([-1, 1]\) is obtained by the reasoning in part a. Thus \( E(x \mid y) = y \), so the decision function \( d(x) = x \) gives an unbiased estimator.

d. Using Equation 4.43, compute

\[
R^2(d, y) = \int_0^1 |d(x) - y|^2 f(x \mid y) \, dx = \int_{y-1}^{y+1} |x - y|^2 (1 - |x - y|) \, dx
\]

\[
= \int_{-1}^{+1} |x|^2(1 - |x|) \, dx = 2 \int_0^1 (|x|^2 - |x|^3) \, dx = \frac{1}{6}.
\]

The third step follows from the substitution \( x \leftarrow x + y \). The risk function \( R(d, y) = 1/\sqrt{6} \) is evidently independent of the ideal value \( y \). \( \square \)