Ma 450: Mathematics for Multimedia

Solution: to Homework Assignment 5

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Due Friday, April 14th, 2017

1. Draw the graphs of \( w(t - 4) \) and \( w(t - \frac{4}{3}) \) on one set of axes for the Haar function \( w(t) \) defined in Equation 5.2.

Solution: The graphs are shown in Figure 1. □

2. For each \((a, b) \in \text{Aff}\), define the linear operator \( \zeta(a, b) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) by

\[
\zeta(a, b)f(t) \overset{\text{def}}{=} \sqrt{a} f(at + b).
\]

(a) Is \( \zeta(a, b) \) unitary for each \((a, b) \in \text{Aff}\)?
(b) Is \((a, b) \mapsto \zeta(a, b)\) a one-to-one mapping on \text{Aff}?  
(c) Is \( \zeta \) a representation of the group \text{Aff}?

Solution: (a) Yes, \( \zeta(a, b) \) is unitary for each \((a, b) \in \text{Aff}\). This may be checked directly, or else by observing that \( \zeta(a, b) = \sigma((a, b)^1) = \sigma((a, b))^1 \), where \( \sigma \) is the

![Figure 1: Graphs of \( w(t - 4) \) and \( w(t - \frac{4}{3}) \) for the Haar function \( w \).](image-url)
unitary representation defined in Eq.5.9 on 137 of our text. Then observe that $T^{-1}$ is unitary whenever $T$ is unitary.

(b) Yes, $(a, b) \mapsto \zeta(a, b)$ is one-to-one. This may be checked directly, or else by observing the relation with faithful representation $\sigma$ as in part a.

(c) Show that $\zeta$ fails to be a representation because it is not multiplicative:

$$
\zeta(a', b')\zeta(a, b)f(t) = \sqrt{a'} [\zeta(a, b)f](a't + b') = \sqrt{ab'} f(a[a't + b'] + b) = \sqrt{a'd} f(aa't + ab' + b) = \zeta(aa', ab' + b)f(t) = \zeta((a, b)(a', b'))f(t).
$$

Thus $\zeta(a', b')\zeta(a, b) = \zeta((a, b)(a', b'))$ when it should be $\zeta(a', b')\zeta(a, b) = \zeta((a', b')(a, b))$. But $\text{Aff}$ is noncommutative: let $(a', b') = (4, 0)$ and $(a, b) = (1, 1)$ to see that $(a', b')(a, b) = (4, 4)$ while $(a, b)(a', b') = (4, 1)$. Choosing $f = 1 = \mathbf{1}_{[0, 1)}$ shows that $\zeta(4, 4)f(t) = 2 \times \mathbf{1}(4t+4)$ while $\zeta(4, 1)f(t) = 2 \times \mathbf{1}(4t+1)$. These functions differ at all $-0.25 < t < 0$, so $\zeta(4, 4) \neq \zeta(4, 1)$. \hfill \Box

3. Let $f = f(a) = f(a, b)$ be the function on $\text{Aff}$ defined by $f(a) = 1_D(a)$, where $1_D$ is the indicator function of the region $D = \{ a = (a, b) : A < a < A', B < b < B' \} \subset \text{Aff}$ for $0 < A < A'$ and $-\infty < B < B' < \infty$. Evaluate $\int_{\text{Aff}} f(a) da$ using the normalized left-invariant integral on $\text{Aff}$.

Solution: Use the integral defined in Equation 5.19:

$$
\int_{\text{Aff}} f(a) da \overset{\text{def}}{=} \int_{b=-\infty}^{\infty} \int_{a=0}^{\infty} f(a, b) \frac{dadb}{a^2} = \int_{b=B}^{B'} \int_{a=A}^{A'} \frac{1}{a^2} dadb = \left( B' - B \right) \int_{a=A}^{A'} \frac{1}{a^2} da = \left( B' - B \right) \left( \frac{1}{A} - \frac{1}{A'} \right).
$$

\hfill \Box

4. Let $w = w(t)$ be the Haar mother function and define

$$
\phi^{J}_{M, K}(t) \overset{\text{def}}{=} \sum_{j=M+1}^{M+J} \frac{1}{2^j} w \left( \frac{t-K}{2^j} \right)
$$

for arbitrary fixed $K \in \mathbb{R}$ and $M, J \in \mathbb{Z}$ with $J > 0$. Prove that

$$
\lim_{J \to \infty} \phi^{J}_{M, K}(t) = 2^{-M} \mathbf{1}_{(K, K+2^M)}(t) \overset{\text{def}}{=} \phi_{M, K}(t),
$$

and also that $\langle \phi^{J}_{M, K}, u \rangle \to \langle \phi_{M, K}, u \rangle$ as $J \to \infty$ for any function $u \in L^2(\mathbb{R})$.  

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Solution: Using \( \phi^J \) as defined in Equation 5.4, evaluate
\[
\phi^J_{M,K}(t) = \sum_{j=M+1}^{M+J} \frac{1}{2^j} w \left( \frac{t - K}{2^j} \right) = \sum_{j=1}^{J} \frac{1}{2^{j+M}} w \left( \frac{t - K}{2^{j+M}} \right)
\]
\[
= \frac{1}{2^M} \sum_{j=1}^{J} \frac{1}{2^j} \left( \frac{1}{2^j} \frac{t - K}{2^M} \right) = \frac{1}{2^M} \phi^J \left( \frac{t - K}{2^M} \right).
\]

Thus for all \( t \in \mathbb{R} \),
\[
\lim_{J \to \infty} \phi^J_{M,K}(t) = \frac{1}{2^M} \lim_{J \to \infty} \phi^J \left( \frac{t - K}{2^M} \right) = \frac{1}{2^M} 1_{\{K,K+2M\}}(t),
\]
as claimed. Lemma 5.1 then gives
\[
\lim_{J \to \infty} \langle \phi^J_{M,K}, u \rangle = \frac{1}{2^M} \int_{-\infty}^{\infty} 1_{\{K,K+2M\}}(t) u(t) dt = \frac{1}{2^M} \int_{K}^{K+2M} u(t) dt = \langle \phi_{M,K}, u \rangle.
\]

\[\Box\]

5. Let \( d \) be a positive integer. Show that the following function has \( d - 1 \) continuous derivatives:
\[
\phi(\xi) = \begin{cases} 
\xi^d e^{-\xi^2}, & \text{if } \xi > 0; \\
0, & \text{if } \xi \leq 0.
\end{cases}
\]

Solution: Calculus techniques suffice to show that \( \phi \) has \( d - 1 \) continuous derivatives everywhere except possibly at the origin. There it must be shown that \( \phi^{(n)}(0+) = 0 \) for \( 0 \leq n < d \). But \( \phi(0+) = 0 \) because \( \xi^d \to 0 \) as \( \xi \to 0^+ \). For the \( n \)th derivative, it is enough to prove that at each \( \xi > 0 \),
\[
\phi^{(n)}(\xi) = \xi^{d-n} P_n(\xi) e^{-\xi},
\]
where \( P_n \) is some polynomial. But this can be established for every \( 0 < n < d \) by induction.

Starting with \( n = 1 \), it is immediate that \( \phi'(\xi) = (d\xi^{d-1} - 2\xi^d)e^{-\xi^2} = \xi^{d-1}(d-2\xi^2)e^{-\xi^2} \) at each \( \xi \neq 0 \). This satisfies the equation with \( P_1(\xi) = d - 2\xi^2 \).

Next, assuming that \( \phi^{(n)} \) satisfies the equation and \( n < d \), use the product rule for derivatives to calculate
\[
\phi^{(n+1)}(\xi) = \left[ (d-n)\xi^{d-n-1} P_n(\xi) + \xi^{d-n} P'_n(\xi) - 2\xi^{d-n+1} P_n(\xi) \right] e^{-\xi^2} = \xi^{d-n-1} P_{n+1}(\xi) e^{-\xi},
\]
where \( P_{n+1}(\xi) = (d-n)P_n(\xi) + \xi P'_n(\xi) - 2\xi^2 P_n(\xi) \). This is of the desired form, as \( P_{n+1} \) is again a polynomial. This completes the induction step.
To complete the solution, note that
\[
\lim_{\xi \to 0^+} \phi^{(n)}(\xi) = P_n(0)e^0 \lim_{\xi \to 0^+} \xi^{d-n} = 0,
\]
for all \(0 \leq n < d\).

\[\Box\]

6. Fix an integer \(d > 0\) and define

\[\phi(\xi) = \begin{cases} 
\exp \left[ - (\log |\xi|)^2d \right], & \text{if } \xi > 0; \\
0, & \text{if } \xi \leq 0.
\end{cases}\]

Show that for any \(n, m \in \mathbb{N}\), we have \(\phi^{(n)}(\xi) = O(1/|\xi|^m)\) as \(\xi \to \pm \infty\).

**Solution:** The result is evident for \(\xi \to -\infty\), as \(\phi^{(n)}(\xi) = 0\) for all \(n \in \mathbb{N}\) and all \(\xi < 0\).

To show that for some positive constant \(C\), \(|\xi|^m \phi^{(n)}(\xi) \leq C\) as \(\xi \to +\infty\), note that modifying the solution to Exercise 5 gives the expression

\[\phi^{(n)}(\xi) = \frac{P(\log |\xi|)}{\xi^n} e^{-(\log |\xi|)^2d},\]

where \(P\) is some polynomial. This can be established for every \(n \geq 0\) by induction. The case \(n = 0\) is immediate, since \(\phi^{(0)}(\xi) = \exp [(\log |\xi|)^2]\) has the stated form, with polynomial \(P = 1\), at each \(\xi \neq 0\).

Next, assuming that \(\phi^{(n)}\) has the stated form for some \(n \geq 0\), calculate

\[\phi^{(n+1)}(\xi) = \frac{d}{d\xi} \phi^{(n)}(\xi) = \frac{P'(\log |\xi|) - 2d(\log |\xi|)^{2d-1} P(\log |\xi|) - nP'(\log |\xi|)}{\xi^{n+1}} \exp \left[ - (\log |\xi|)^2d \right],\]

which is of the desired form with a new polynomial \(Q(z) = P'(z) - 2z^{2d-1}P(z) - nP(z)\).

This completes the induction step and the proof of the formula.

Using the formula, evaluate

\[\xi^m \phi^{(n)}(\xi) = \xi^m \frac{P(\log \xi)}{\xi^n} \exp \left[ - (\log \xi)^2d \right] = \xi^{m-n-(\log \xi)^{2d-1}}P(\log \xi),\]

where \(P\) is some polynomial depending on \(n\). But this expression will tend to zero as \(\xi \to \infty\), since the exponent \(m - n - (\log \xi)^{2d-1}\) will become more and more negative as \(\xi \to \infty\), eventually cancelling the slow growth from \(P(\log \xi)\).\[\Box\]
7. Compute \( \|w\| \), where
\[
\mathcal{F}w(\xi) = \begin{cases} 
  e^{-(\log \xi)^2}, & \text{if } \xi > 0; \\
  0, & \text{if } \xi \leq 0.
\end{cases}
\]

(Hint: use Plancherel’s theorem and Equation B.6 in Appendix B.)

**Solution:** By Plancherel’s theorem, \( \|w\| = \|\mathcal{F}w\| \). Substitute \( \xi \leftarrow e^{\eta + \frac{1}{4}} \) to compute
\[
\|\mathcal{F}w\|^2 = \int_{\xi=0}^{\infty} e^{-2(\log \xi)^2} d\xi = \int_{\eta=-\infty}^{\infty} e^{-2(\eta + \frac{1}{4})^2} e^{\eta + \frac{1}{4}} d\eta = e^{1/8} \int_{\eta=-\infty}^{\infty} e^{-2\eta^2} d\eta.
\]

Finally, substitute \( x \leftarrow \eta \sqrt{\frac{2}{\pi}} \) into Equation B.6, \( \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1 \), to get \( \int_{-\infty}^{\infty} e^{-2\eta^2} d\eta = \sqrt{\frac{\pi}{2}} \). Thus \( \|w\| = \sqrt{\|\mathcal{F}w\|^2} = e^{1/16} \left( \frac{\pi}{2} \right)^{1/4} \approx 1.1917. \)

8. Let \( w \) be the function defined by
\[
\mathcal{F}w(\xi) = \begin{cases} 
  e^{-(\log |\xi|)^2}, & \text{if } \xi \neq 0; \\
  0, & \text{if } \xi = 0.
\end{cases}
\]

Show that \( w \) is admissible and compute its normalization constant \( c_w \).

**Solution:** First note that \( \mathcal{F}w(-\xi) = \mathcal{F}w(\xi) \), so \( |\mathcal{F}w(-\xi)|^2 = |\mathcal{F}w(\xi)|^2 \). Thus if the \( +\xi \) admissibility integral exists, then the \( -\xi \) integral exists as well and has the same value.

Next, compute the \( +\xi \) admissibility integral:
\[
c_w = \int_{0}^{\infty} \frac{|\mathcal{F}w(\xi)|^2}{\xi} d\xi = \int_{\xi=0}^{\infty} e^{-2(\log \xi)^2} |\xi| d\xi = \int_{\eta=-\infty}^{\infty} e^{-2\eta^2} e^{\eta} d\eta = \int_{\eta=-\infty}^{\infty} e^{-2\eta^2} d\eta.
\]

This follows from the substitution \( \xi \leftarrow e^{\eta} \). Finally, substitute \( x \leftarrow \eta \sqrt{\frac{2}{\pi}} \) into Equation B.6, \( \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1 \), to get \( c_w = \int_{-\infty}^{\infty} e^{-2\eta^2} d\eta = \sqrt{\frac{\pi}{2}} \approx 1.2533 \). Thus \( w \) is admissible.

9. Fix \( A < 0, B > 0, \) and \( R > 1 \) and suppose that \( w = w(x) \) is a function satisfying \( \mathcal{F}w(\xi) = 1 \) if \( RA < \xi < A \) or \( B < \xi < RB \), with \( \mathcal{F}w(\xi) = 0 \), otherwise. Show that \( w \) satisfies the admissibility condition of Theorem 5.2, and compute the normalization constant \( c_w \). Give a formula for \( w \).

**Solution:** Plancherel’s theorem guarantees that \( w \) belongs to \( L^2(\mathbb{R}) \), since \( \|w\| = \|\mathcal{F}w\| = \sqrt{(B-A)(R-1)} < \infty \).
Compute the two admissibility integrals:
\[
\int_0^\infty \frac{|\mathcal{F}w(-\xi)|^2}{\xi} \, d\xi = \int_{-A}^{-RA} \frac{1}{\xi} \, d\xi = \log(-RA) - \log(-A) = \log R;
\]
\[
\int_0^\infty \frac{|\mathcal{F}w(\xi)|^2}{\xi} \, d\xi = \int_{B}^{RB} \frac{1}{\xi} \, d\xi = \log(RB) - \log(B) = \log R.
\]
These are finite and equal for \( R > 1 \), so \( w \) is admissible with normalization constant \( c_w = \log R \).

The inverse Fourier integral transform of \( \mathcal{F}w \) gives \( w \):
\[
w(x) = \int_{-\infty}^{\infty} e^{2\pi i x \xi} \mathcal{F}w(\xi) \, d\xi = \int_{RA}^{RB} e^{2\pi i x \xi} \, d\xi = \frac{e^{2\pi i x A} - e^{2\pi i x RA} + e^{2\pi i x RB} - e^{2\pi i x B}}{2\pi i x}.
\]

10. Show that if \( h = \{h(k) : k \in \mathbb{Z}\} \) is a self-orthonormal filter, and \( M \) is any fixed integer, then the sequence defined by
\[
g(k) = (-1)^k h(2M - 1 - k), \quad \text{for all } k \in \mathbb{Z},
\]
satisfies the completeness condition of Equation 5.45.

**Solution:** Check Equation 5.45 using the formula with shift \( 2M \):
\[
\sum_k h(2k + n)h(2k + m) + \sum_k g(2k + n)g(2k + m) = \delta(n - m).
\]
This can be shown case-by-case. We first write \( g \) in terms of \( h \), making the sum
\[
\sum_k h(2k + n)h(2k + m) + (-1)^{n+m} \sum_k h(2k + 2M - 1 - n)h(2k + 2M - 1 - m).
\]
The second sum has been simplified with \( k \leftarrow -k \) and the observation that \((-1)^{n+2k+m+2k} = (-1)^{n+m} \) for all \( k \in \mathbb{Z} \).
Then we put \( p = m - n \) to have \( n + m = 2n + p \) and \((-1)^{n+m} = (-1)^p \) and consider the cases:
- If \( n = 2n' \) is even, then substituting \( k \leftarrow k - n' \) in the first sum and \( k \leftarrow k + n' - M \) in the second reduces them to
\[
\sum_k h(2k)h(2k + p) + (-1)^p \sum_k h(2k - 1)h(2k - 1 - p).
\]
- If \( p = 2p' \) is even, then substituting \( k \leftarrow k + p' \) in the second sum gives
\[
\sum_k h(2k)h(2k + p) + \sum_k h(2k - 1 + p)h(2k - 1) = \sum_k h(k)h(k + p)
\]
\[
= \sum_k h(k)h(k + 2p') = \delta(p') = \delta(n - m).
\]

- If \( p = 2p' - 1 \) is odd, then substituting \( k \leftarrow k + p' \) in the second sum gives
\[
\sum_k h(2k)h(2k + p) - \sum_k h(2k + p)h(2k) = 0.
\]
This agrees with the value of \( \delta(n - m) \), which is 0 in this case since \( p = m - n \) being odd means \( n \neq m \).

- If \( n = 2n' + 1 \) is odd, then substituting \( k \leftarrow k - n' \) in the first sum and \( k \leftarrow k + n' + 1 - M \) in the second reduces them to
\[
\sum_k h(2k + 1)h(2k + 1 + p) + (-1)^p \sum_k h(2k)h(2k - p).
\]

- If \( p = 2p' \) is even, then substituting \( k \leftarrow k + p' \) in the second sum gives
\[
\sum_k h(2k + 1)h(2k + 1 + p) + \sum_k h(2k + p)h(2k) = \sum_k h(k)h(k + p)
\]
\[
= \sum_k h(k)h(k + 2p') = \delta(p') = \delta(n - m).
\]

- If \( p = 2p' - 1 \) is odd, then substituting \( k \leftarrow k + p' \) in the second sum gives
\[
\sum_k h(2k + 1)h(2k + 1 + p) - \sum_k h(2k + p + 1)h(2k + 1) = 0.
\]
This agrees with the value of \( \delta(n - m) \), which is 0 in this case since \( p = m - n \) being odd means \( n \neq m \).

\( \square \)

11. a. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the antisymmetry condition \( h(0) = -h(3) \) and \( h(1) = -h(2) \)?

b. Are there any real-valued orthogonal low-pass CQFs of length 4 satisfying the symmetry condition \( h(0) = h(3) \) and \( h(1) = h(2) \)?

**Solution:** a. No. Antisymmetry would violate the normalization condition \( h(0) + h(2) = \frac{1}{\sqrt{2}} = h(1) + h(3) \). Thus no antisymmetric real-valued orthogonal low-pass CQFs of length 4 exist.
b. No. Let \( h \) be an orthogonal CQF with nonzero real coefficients \( h(0), h(1), h(2), \) and \( h(3) \). Then \( h \) must be of the form

\[
\begin{align*}
    h(0) &= \frac{1 - c}{\sqrt{2(1 + c^2)}}; \\
    h(1) &= \frac{1 + c}{\sqrt{2(1 + c^2)}}; \\
    h(2) &= \frac{c(c + 1)}{\sqrt{2(1 + c^2)}}; \\
    h(3) &= \frac{c(c - 1)}{\sqrt{2(1 + c^2)}},
\end{align*}
\]

where \( c \) is some real number different from 0 and \( \pm 1 \). The symmetry conditions imply \( 1 - c = c(c - 1) \) and \( 1 + c = c(c + 1) \), which implies \( c^2 = 1 \) and thus \( c = \pm 1 \). But that value of \( c \) leads to a filter of length 2, the Haar filter. So no symmetric real-valued orthogonal CQFs of length 4 exist. \( \square \)

12. Suppose that an orthogonal MRA has a scaling function \( \phi \) satisfying \( \phi(t) = 0 \) for \( t \notin [a,b] \). Prove that the low-pass filter \( h \) for this MRA must satisfy \( h(n) = 0 \) for all \( n \notin [2a - b, 2b - a] \). (This makes explicit the finite support of \( h \) in Equation 5.36.)

**Solution:** If \( \phi(t) = 0 \) for \( t \notin [a,b] \), then \( \phi(2t - n) = 0 \) for \( t \notin \left[ \frac{a+n}{2}, \frac{b+n}{2} \right] \). Use the orthonormality of \( \{ \sqrt{2}\phi(2t - k) : k \in \mathbb{Z} \} \) in \( V_{-1} \) to compute

\[
\frac{1}{\sqrt{2}} h(n) = \left\langle \phi(2t - n), \sum_k h(k) \sqrt{2} \phi(2t - k) \right\rangle = \left\langle \phi(2t - n), \phi(t) \right\rangle.
\]

But the support intervals \( \left[ \frac{a+n}{2}, \frac{b+n}{2} \right] \) and \( [a,b] \) of the two factors in the inner product will not overlap if \( (a+n)/2 > b \iff n > 2b - a \) or if \( (b + n)/2 < a \iff n < 2a - b \). Thus \( h(n) = 0 \) if \( n \notin [2a - b, 2b - a] \). \( \square \)

13. Suppose that \( h = \{ h(k) : k \in \mathbb{Z} \} \) and \( g = \{ g(k) : k \in \mathbb{Z} \} \) satisfy the orthogonal CQF conditions. Show that the 2-periodizations \( h_2, g_2 \) of \( h \) and \( g \) are the Haar filters. Namely, show that \( h_2(0) = h_2(1) = g_2(0) = -g_2(1) = 1/\sqrt{2} \).

**Solution:** Use the normalization conditions for \( h \) and \( g \) to evaluate the 2-periodization formula:

\[
\begin{align*}
    h_2(0) &= \sum_k h(0 + 2k) = 1/\sqrt{2}; \\
    h_2(1) &= \sum_k h(1 + 2k) = 1/\sqrt{2}; \\
    g_2(0) &= \sum_k g(0 + 2k) = 1/\sqrt{2}; \\
    g_2(1) &= \sum_k g(1 + 2k) = -1/\sqrt{2}.
\end{align*}
\]

Since \( h_2 \) and \( g_2 \) are 2-periodic, this determines all their values. \( \square \)

14. Let \( \phi \) be the scaling function of an orthogonal MRA, and let \( \psi \) be the associated mother function. For \( (x,y) \in \mathbb{R}^2 \), define

\[
\begin{align*}
    e_0(x,y) &= \phi(x)\phi(y), & e_1(x,y) &= \phi(x)\psi(y) \\
    e_2(x,y) &= \psi(x)\phi(y), & e_3(x,y) &= \psi(x)\psi(y).
\end{align*}
\]
Prove that the functions \( \{e_n : n = 0, 1, 2, 3\} \) are orthonormal in \( L^2(\mathbb{R}^2) \), the inner product space of square-integrable functions on \( \mathbb{R}^2 \).

**Solution:** First note that \( \phi \) and \( \psi \) are compactly supported, so \( e_0, e_1, e_2, e_3 \) vanish outside some closed and bounded rectangles in \( \mathbb{R}^2 \). Also, both \( \phi \) and \( \psi \) are integrable and square-integrable as functions of one variable, so \( e_0, e_1, e_2, e_3 \) are integrable by iterating one-dimensional integrals. Write \( e_i(x) \) for the \( x \)-dependent factor of \( e_i(x, y) \) and \( e_i(y) \) for the \( y \)-dependent factor of \( e_i(x, y) \); then \( e_0 = e_1 = e_2 = e_3 = \phi \), while \( e_1 = e_2 = e_3 = \psi \). Thus the inner products in \( L^2(\mathbb{R}^2) \) are computable as follows:

\[
\langle e_i, e_j \rangle = \int_{\mathbb{R}^2} e_i(x, y)e_j(x, y)\,dxdy
= \left( \int_{\mathbb{R}} e_i^1(x)e_j^1(x)\,dx \right) \left( \int_{\mathbb{R}} e_i^2(y)e_j^2(y)\,dy \right)
= \langle e_i^1, e_j^1 \rangle \langle e_i^2, e_j^2 \rangle, \quad i, j \in \{0, 1, 2, 3\}.
\]

But if \( i \neq j \), then at least one of these inner products is \( \langle \phi, \psi \rangle \) or \( \langle \psi, \phi \rangle \), which are both zero since \( \phi \perp \psi \). Hence \( \{e_0, e_1, e_2, e_3\} \) is an orthogonal set in \( L^2(\mathbb{R}^2) \).

On the other hand, if \( i = j \), then the two inner products are both 1 since \( \|\phi\| = \|\psi\| = 1 \). Hence \( \{e_0, e_1, e_2, e_3\} \) is an orthonormal set in \( L^2(\mathbb{R}^2) \). \( \square \)