Math 450: Mathematics for Multimedia Midterm Examination

Name: MODEL SOLUTIONS

Friday, 11 March 2011

8 problems on 8 pages

No materials other than this test, a pen or pencil, and your textbook are permitted. Please write your complete answers in the space provided.

1. Suppose that a and b are relatively prime integers, and also that c and d are relatively prime integers. Must ac and bd be relatively prime?

   Solution: No. Take $a = 5$, $b = 1$, $c = 1$, and $d = 5$. Then $ac = 5 = bd$ share the common factor 5. □

2. Express the number $x = 3.3333\ldots$ (base 16) as a decimal expansion in base 10.

   Solution: Write

$$x = 3 + \sum_{k=1}^{\infty} \frac{3}{16^k} = 3 + \frac{3}{16} \sum_{k=0}^{\infty} \frac{1}{16^k} = 3 + \frac{3}{16} \frac{1}{1 - 1/16} = 3 + \frac{1}{5} = 3.2.$$

□

3. Suppose that $\mathbf{x}, \mathbf{y}$ are vectors in an inner product space $\mathbf{X}$, with $\|\mathbf{x}\| = 5$ and $\|\mathbf{y}\| = 2$.

   (a) What is the maximum possible value of $\|\mathbf{x} + \mathbf{y}\|$?

   (b) What is the minimum possible value of $\|\mathbf{x} + \mathbf{y}\|$?

   (c) What is the maximum possible value of $\langle \mathbf{x}, \mathbf{y} \rangle$?

   (d) What is the minimum possible value of $\langle \mathbf{x}, \mathbf{y} \rangle$?

   (e) What is the minimum possible value of $| \langle \mathbf{x}, \mathbf{y} \rangle |$?

   Solution: a. By the triangle inequality, or the norm sublinearity axiom, $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\| = 7$. This is achieved in the example space $\mathbf{X} = \mathbf{R}^2$ with the vectors $\mathbf{x} = 5\mathbf{e}_1$ and $\mathbf{y} = 2\mathbf{e}_1$, so no smaller value can be the maximum.

b. By the triangle inequality, or the norm sublinearity axiom, $\|\mathbf{x}+\mathbf{y}\| \geq |\|\mathbf{x}\|-\|\mathbf{y}\|| = 3$. This is achieved in the example space $\mathbf{X} = \mathbf{R}^2$ with the vectors $\mathbf{x} = 5\mathbf{e}_1$ and $\mathbf{y} = -2\mathbf{e}_1$, so no larger value can be the minimum.

c. By the Cauchy-Schwarz inequality, $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\| = 10$. Hence $\langle \mathbf{x}, \mathbf{y}\rangle \leq= 10$. This is achieved in the example space $\mathbf{X} = \mathbf{R}^2$ with the vectors $\mathbf{x} = 5\mathbf{e}_1$ and $\mathbf{y} = 2\mathbf{e}_1$, so no smaller value can be the maximum.

d. By the Cauchy-Schwarz inequality, $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\| = 10$. Hence $\langle \mathbf{x}, \mathbf{y}\rangle \geq= -10$. This is achieved in the example space $\mathbf{X} = \mathbf{R}^2$ with the vectors $\mathbf{x} = 5\mathbf{e}_1$ and $\mathbf{y} = -2\mathbf{e}_1$, so no larger value can be the minimum.

e. By definition, $|\langle \mathbf{x}, \mathbf{y}\rangle| \geq 0$. This minimum is achieved in the example space $\mathbf{X} = \mathbf{R}^2$ with orthogonal vectors $\mathbf{x}, \mathbf{y}$ such as $\mathbf{x} = 3\mathbf{e}_1$ and $\mathbf{y} = 2\mathbf{e}_2$, so no larger value can be the minimum. □

4. Find the biorthogonal dual of the basis $E = \{e_n(x) \stackrel{\text{def}}{=} e^{2\pi i n x} : n \in \mathbf{Z}\}$ of $\mathbf{Lip}([0,1])$.

Solution: Since E is already an orthonormal basis for **Lip**, it is its own biorthogonal dual. □

5. Suppose that $A \in \mathbf{Mat}(N \times N)$ satisfies $\operatorname{tr}(A^n) = 0$ for all $n = 1, 2, 3, \ldots$.

(a) Must $A = 0$?

(b) Suppose in addition that A is symmetric: $A^* = A$. Must $A = 0$?

Solution: (a) No. Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$, which nonetheless satisfies $\operatorname{tr}(A^n) = 0$ for all $n \in \mathbf{Z}^+$.

(b) Yes. Since $A^* = A$ we have $A^2 = A^*A$, so $0 = \operatorname{tr}(A^*A) = \|A\|_{\text{HS}}^2$ implies that $A = 0$ by the nondegeneracy of $\|\cdot\|_{\text{HS}}$. □

6. Let $T : \mathbf{R}^2 \to \mathbf{R}^4$ be the linear transformation defined by $T(x, y) = (x, y, x+y, x-y)$.

(a) Compute T^* with respect to the usual inner products.

(b) Compute $\|T\|_{\text{op}}$ with respect to the usual Euclidean norms.

Solution: (a) For all $(x, y) \in \mathbf{R}^2$ and all $(p, q, r, s) \in \mathbf{R}^4$, we have

$$\begin{aligned} \langle (p,q,r,s), T(x,y)\rangle_{\mathbf{R}^4} &= px + qy + r(x+y) + s(x+y) \\ &= (p+r+s)x + (q+r-s)y \;=\; \langle T^*(p,q,r,s), (x,y)\rangle_{\mathbf{R}^2}. \end{aligned}$$

Thus, by nondegeneracy of the inner products, we conclude that

$$T^*(p,q,r,s) = (p+r+s, q+r-s).$$

This may also be obtained by transposing a matrix representing T with respect to the standard bases for $\mathbf{R}^2$ and $\mathbf{R}^4$.

(b)Compute

$$\|T(x,y)\|^2 = x^2 + y^2 + (x+y)^2 + (x-y)^2 = 3[x^2+y^2] = 3\|(x,y)\|^2.$$

Thus $\|T\|_{\text{op}} = \sqrt{3}$. □

7. Find the complex exponential Fourier series of the 1-periodic Hartley basis function $\operatorname{cas}(2\pi kt)$, where $k > 0$ is an integer.

Solution: Note that $\cos(2\pi kt) = \frac{1}{2}\left(e^{2\pi ikt} + e^{-2\pi ikt}\right)$.

Likewise, $\sin(2\pi kt) = \frac{1}{2i}\left(e^{2\pi ikt} - e^{-2\pi ikt}\right)$.

Hence the complex Fourier series of the sum of these two functions has $c(k) = \frac{1}{2} + \frac{1}{2i} = \frac{1-i}{2}$ and $c(-k) = \frac{1}{2} - \frac{1}{2i} = \frac{1+i}{2}$, with $c(n) = 0$ for all $n \neq \pm k$. □

8. Suppose that ϕ has Fourier integral transform $\mathcal{F}\phi$. Fix $a \in \mathbf{R}$, let $\phi_a(x) \stackrel{\text{def}}{=} e^{2\pi ax}\phi(x)$, and compute $\mathcal{F}\phi_a$ in terms of $\mathcal{F}\phi$.

Solution: Evaluate the integral:

$$\mathcal{F}\phi_a(\xi) = \int_{-\infty}^{\infty} e^{-2\pi ix\xi} e^{2\pi ax}\,\phi(x)\,dx = \int_{-\infty}^{\infty} e^{-2\pi iy(\xi+ia)}\,\phi(y)\,dy = \mathcal{F}\phi(\xi+ia).$$

□