Please do Exercises 1, 4*, 7, 9*, 11, 12, 13, 16, 21*, 26, 28, 31, 32, 33, 36, 37.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: “textbook” refers to “Real Analysis for Graduate Students,” version 2.1, by Richard F. Bass. These exercises originate from that source.

1. Suppose \((X, \mathcal{A})\) is a measurable space, \(f : X \to \mathbb{R}\) is a function, and \(\{ x : f(x) > r \} \in \mathcal{A}\) for all \(r \in \mathbb{Q}\). Prove that \(f\) is measurable.

2. Let \(f : (0, 1) \to \mathbb{R}\) be such that for every \(x \in (0, 1)\) there exist \(r > 0\) and a Borel measurable function \(g\), both depending on \(x\), such that \(f\) and \(g\) agree on \((x - r, x + r) \cap (0, 1)\). Prove that \(f\) is Borel measurable.

3. Suppose \(f_n\) is a sequence of measurable functions. Prove that

\[ A = \{ x : \lim_{n \to \infty} f_n(x) \text{ exists} \} \]

is a measurable set.

4. If \(f : \mathbb{R} \to \mathbb{R}\) is Lebesgue measurable, prove that there exists a Borel measurable function \(g\) such that \(f = g\) a.e.

5. Give an example of a collection of measurable non-negative functions \(\{f_\alpha\}_{\alpha \in A}\) such that if \(g\) is defined by \(g(x) = \sup_{\alpha \in A} f_\alpha(x)\), then \(g\) is finite for all \(x\) but \(g\) is non-measurable. (Hint: \(A\) is allowed to be uncountable.)
6. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is Lebesgue measurable and \( g : \mathbb{R} \to \mathbb{R} \) is continuous. Prove that \( g \circ f \) is Lebesgue measurable. Is this true if \( g \) is Borel measurable instead of continuous? Is this true if \( g \) is Lebesgue measurable instead of continuous?

7. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is Borel measurable. Define \( A \) to be the smallest \( \sigma \)-algebra containing the sets \( \{ x : f(x) > a \} \) for every \( a \in \mathbb{R} \). Suppose \( g \) is measurable with respect to \( A \), namely that \((\forall a \in \mathbb{R})\{ x : g(x) > a \} \in A\). Prove that there exists a Borel measurable function \( h : \mathbb{R} \to \mathbb{R} \) such that \( g = h \circ f \).

8. It is known that there exist discontinuous real-valued functions \( f \) such that
\[
(\forall x, y \in \mathbb{R}) \ f(x + y) = f(x) + f(y).
\]
(An example may be constructed using Zorn’s lemma.) Prove that if \( f \) satisfies (1) and in addition \( f \) is Lebesgue measurable, then \( f \) is continuous.

9. Verify Equation (6.5) on textbook p.48. Namely, for measure space \((X, \mathcal{A}, \mu)\), show that if \( \sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{j=1}^{m} b_j \chi_{B_j} \) for \( A_i, B_j \in \mathcal{A} \) and \( a_i, b_j \in \mathbb{R} \), then
\[
\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{j=1}^{m} b_j \mu(B_j).
\]

10. Suppose \( f \) is non-negative and measurable and \( \mu \) is \( \sigma \)-finite. Show there exist simple functions \( s_n \) increasing to \( f \) at each point such that \( \mu(\{ x : s_n(x) \neq 0 \}) < \infty \) for each \( n \).

11. Let \( f \) be a non-negative measurable function. Prove that
\[
\lim_{n \to \infty} \int \min(f, n) = \int f.
\]

12. Let \((X, \mathcal{A}, \mu)\) be a measure space and suppose \( \mu \) is \( \sigma \)-finite. Suppose \( f \) is integrable. Prove that given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\int_{A} |f(x)| \mu(dx) < \epsilon
\]
whenever \( \mu(A) < \delta \).

13. Suppose \( \mu(X) < \infty \) and \( f_n \) is a sequence of bounded real-valued measurable functions that converge to \( f \) uniformly. Prove that
\[
\int f_n \, d\mu \to \int f \, d\mu.
\]
This is sometimes called the *bounded convergence theorem*. (Hint: prove this with or without the dominated convergence theorem.)

14. If $f_n$ is a sequence of non-negative integrable functions such that $f_n(x)$ decreases to $f(x)$ for every $x$, prove that $\int f_n \, d\mu \to \int f \, d\mu$.

15. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and suppose $f$ is a non-negative, measurable function that is finite at each point of $X$, but is not necessarily integrable. Prove that there exists a continuous increasing function $g : [0, \infty) \to [0, \infty)$ such that $\lim_{x \to \infty} g(x) = \infty$ and $g \circ f$ is integrable.

16. State and prove a version of the Dominated Convergence Theorem for complex-valued functions. (Hint: prove this as a corollary to the dominated convergence theorem for real-valued functions.)

17. Suppose $f_n, g_n, f, g$ are integrable, $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \leq g_n$ for each $n$, and $\int g_n \to \int g$. Prove that $\int f_n \to \int f$. (Hint: use the dominated convergence theorem.)

18. Give an example of a sequence of non-negative functions $f_n$ tending to 0 pointwise such that $\int f_n \to 0$ but there is no integrable function $g$ such that $f_n \leq g$ for all $n$.

19. Suppose $(X, \mathcal{A}, \mu)$ is a measure space, $f$ and each $f_n$ is integrable and non-negative, $f_n \to f$ a.e, and $\int f_n \to \int f$. Prove that for each $A \in \mathcal{A}$,

$$\int_A f_n \, d\mu \to \int_A f \, d\mu.$$
23. Find the limit
\[ \lim_{n \to \infty} \int_0^n \left( 1 + \frac{x}{n} \right)^{-n} \log(2 + \cos(x/n)) \, dx \]
and justify your reasoning.

24. Find the limit
\[ \lim_{n \to \infty} \int_0^n \left( 1 - \frac{x}{n} \right)^n \log(2 + \cos(x/n)) \, dx \]
and justify your reasoning.

25. Prove that the limit exists and find its value:
\[ \lim_{n \to \infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} \log(2 + \cos(x/n)) \, dx \]

26. Prove that the limit exists and determine its value:
\[ \lim_{n \to \infty} \int_0^\infty ne^{nx} \sin(1/x) \, dx \]

27. Let \( g : \mathbb{R} \to \mathbb{R} \) be integrable and let \( f : \mathbb{R} \to \mathbb{R} \) be bounded, measurable, and continuous at 1. Prove that
\[ \lim_{n \to \infty} \int_{-n}^n f \left( 1 + \frac{x}{n^2} \right) g(x) \, dx \]
evaluates and determine its value.

28. Suppose \( \mu(X) < \infty \), \( f_n \) converges to \( f \) uniformly, and each \( f_n \) is integrable. Prove that \( f \) is integrable and \( \int f_n \to \int f \). Is the condition \( \mu(X) < \infty \) necessary?

29. Prove that
\[ \sum_{k=1}^{\infty} \frac{1}{(p+k)^2} = -\int_0^1 \frac{x^p}{1-x} \log x \, dx \]
for \( p > 0 \). (Hint: use the fundamental theorem of calculus, to be proved in textbook Chapter 8, namely if \( f \) is continuous on \([a,b]\) and \( F \) is differentiable on \([a,b]\) with derivative \( f \), then \( \int_a^b f(x) \, dx = F(b) - F(a) \).)

30. Let \( f_n \) be a sequence of measurable real-valued functions on \([0,1]\) that is uniformly bounded.
   a. Show that if \( A \) is a Borel subset of \([0,1]\), then there exists a subsequence \( n_j \) such that \( \int_A f_{n_j}(x) \, dx \) converges.
b. Show that if \( \{A_i\} \) is a countable collection of Borel subsets of \([0, 1]\), then there exists a subsequence \( n_j \) such that \( \int_{A_i} f_{n_j}(x) \,dx \) converges for each \( i \).

c. Show that there exists a subsequence \( n_j \) such that \( \int_A f_{n_j}(x) \,dx \) converges for each Borel subset \( A \) of \([0, 1]\).

31. Let \((X, \mathcal{A}, \mu)\) be a measure space. A sequence of measurable functions \( \{f_n\} \) is uniformly integrable if, given \( \epsilon > 0 \), there exists \( M \) such that
\[
\int_{\{x: |f_n(x)| > M\}} |f_n(x)| \,d\mu < \epsilon
\]
for each \( n \). The sequence is uniformly absolutely continuous if, given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\left| \int_A f_n \,d\mu \right| < \epsilon
\]
for each \( n \) and each \( A \in \mathcal{A} \) with \( \mu(A) < \delta \).

Suppose that \( \mu \) is a finite measure. Prove that \( \{f_n\} \) is uniformly integrable if and only if \( \sup_n \int |f_n| \,d\mu < \infty \) and \( \{f_n\} \) is uniformly absolutely continuous.

32. Suppose \( \mu \) is a finite measure, \( f_n \to f \) a.e., and \( \{f_n\} \) is uniformly integrable (see Exercise 31). Prove that \( \int |f_n - f| \to 0 \). (This is known as the Vitali convergence theorem.)

33. Suppose \( \mu \) is a finite measure, \( f_n \to f \) a.e., each \( f_n \) is integrable, and \( \int |f_n - f| \to 0 \). Prove that \( \{f_n\} \) is uniformly integrable (see Exercise 31).

34. Suppose \( \mu \) is a finite measure and for some \( \epsilon > 0 \),
\[
\sup_n \int |f_n|^{1+\epsilon} \,d\mu < \infty.
\]
Prove that \( \{f_n\} \) is uniformly integrable (see Exercise 31).

35. Suppose \( \{f_n\} \) is a uniformly integrable sequence of functions defined on \([0, 1]\). Prove that there is a subsequence \( n_j \) such that \( \int_0^1 f_{n_j} g \,dx \) converges whenever \( g \) is a real-valued bounded measurable function.

36. Suppose \( \mu_n \) is a sequence of measures on \((X, \mathcal{A})\) such that \( \mu_n(X) = 1 \) for all \( n \) and \( \mu_n(A) \) converges as \( n \to \infty \) for each \( A \in \mathcal{A} \). Call the limit \( \mu(A) \).

a. Prove that \( \mu \) is a measure.

b. Prove that \( \int f \,d\mu_n \to \int f \,d\mu \) whenever \( f \) is bounded and measurable.
c. Prove that
\[ \int f \, d\mu \leq \liminf_{n \to \infty} \int f \, d\mu_n \]
whenever \( f \) is non-negative and measurable.

37. Let \((X, \mathcal{A}, \mu)\) be a measure space and let \(f\) be non-negative and integrable. Define \(\nu\) on \(\mathcal{A}\) by
\[ \nu(A) \overset{\text{def}}{=} \int_A f \, d\mu. \]

a. Prove that \(\nu\) is a measure.

b. Prove that if \(g\) is integrable with respect to \(\nu\), then \(fg\) is integrable with respect to \(\mu\) and
\[ \int g \, d\nu = \int fg \, d\mu. \]

38. Suppose \(\mu\) and \(\nu\) are finite positive measures on the Borel \(\sigma\)-algebra on \([0,1]\) such that
\[ \int f \, d\mu = \int f \, d\nu \]
whenever \(f\) is real-valued and continuous on \([0,1]\). Prove that \(\mu = \nu\).

39. Let \(\mathcal{B}\) be the Borel \(\sigma\)-algebra on \([0,1]\). Let \(\mu_n\) be a sequence of finite measures on \(([0,1], \mathcal{B})\) and let \(\mu\) be another finite measure on \(([0,1], \mathcal{B})\). Suppose \(\mu_n([0,1]) \to \mu([0,1])\). Prove that the following are equivalent:

a. \(\int f \, d\mu_n \to \int f \, d\mu\) whenever \(f\) is a continuous real-valued function on \([0,1]\);

b. \(\limsup_{n \to \infty} \mu_n(F) \leq \mu(F)\) for all closed \(F \subset [0,1]\);

c. \(\liminf_{n \to \infty} \mu_n(G) \geq \mu(G)\) for all open \(G \subset [0,1]\);

d. \(\lim_{n \to \infty} \mu_n(A) = \mu(A)\) for all \(A \in \mathcal{B}\) such that \(\mu(\partial A) = 0\), where \(\partial A \overset{\text{def}}{=} \overline{A} - A^o\) is the boundary of \(A\);

e. \(\lim_{n \to \infty} \mu_n([0, x]) = \mu([0, x])\) for all \(x \in [0,1]\) such that \(\mu(\{x\}) = 0\).

40. Let \(\mathcal{B}\) be the Borel \(\sigma\)-algebra on \([0,1]\). Suppose \(\mu_n\) are finite measures on \(([0,1], \mathcal{B})\) such that \(\int f \, d\mu_n \to \int_0^1 f(x) \, dx\) whenever \(f\) is a real-valued continuous function on \([0,1]\). Suppose that \(g\) is a bounded measurable function such that the set of discontinuities of \(g\) has measure 0. prove that
\[ \int g \, d\mu_n \to \int_0^1 g(x) \, dx. \]

41. Let \(\mathcal{B}\) be the Borel \(\sigma\)-algebra on \([0,1]\). Let \(\mu_n\) be a sequence of finite measures on \(([0,1], \mathcal{B})\) such that \(\sup_n \mu_n([0,1]) < \infty\). Define \(\alpha_n(x) = \mu_n([0, x])\).
a. If \( r \in [0, 1] \) is rational, prove that there exists a subsequence \( \{n_j\} \) such that \( \alpha_{n_j}(r) \) converges.

b. Prove that there exists a subsequence \( \{n_j\} \) such that \( \alpha_{n_j}(r) \) converges for every rational in \([0, 1]\).

c. Let \( \tilde{\alpha}(r) \overset{\text{def}}{=} \lim_{n \to \infty} \alpha_n(r) \) for rational \( r \in [0, 1] \). Prove that \( r > s \) with \( r, s \in \mathbb{Q} \cap [0, 1] \) implies \( \tilde{\alpha}(r) \leq \tilde{\alpha}(s) \).

d. Define

\[
\alpha(x) = \lim_{r \to x^+, r \in \mathbb{Q}} \tilde{\alpha}(r).
\]

Prove that

\[
\alpha(x) = \inf \{ \tilde{\alpha}(r) : r > x, r \in \mathbb{Q} \cap [0, 1] \}.
\]

e. Let \( \mu \) be the Lebesgue-Stieltjes measure associated with \( \alpha \). Prove that

\[
\int f \, d\mu_n \to \int f \, d\mu
\]

whenever \( f \) is a continuous real-valued function on \([0, 1]\).