1. Show that if $f$ and $g$ are absolutely continuous on an interval $[a, b]$, then so is their product $fg$.

2. Suppose that $f$ and $g$ are absolutely continuous functions on an interval $[a, b]$. Prove the integration by parts formula:

$$f(b)g(b) - f(a)g(a) = \int_a^b f(x)g'(x) \, dx + \int_a^b f'(x)g(x) \, dx.$$ 

3. Prove that if $f$ is integrable and real-valued, $a \in \mathbb{R}$, and

$$F(x) \overset{\text{def}}{=} \int_a^x f(y) \, dy,$$

then $F$ is of bounded variation and is absolutely continuous.

4. Suppose that $f$ is a real-valued continuous function on $[0, 1]$ and that $\epsilon > 0$. Prove that there exists a continuous function $g$ such that $g'(x)$ exists and equals 0 for a.e. $x$ and

$$\sup_{x \in [0,1]} |f(x) - g(x)| < \epsilon.$$
5. Suppose $f$ is a real-valued continuous function on $[0, 1]$ and $f$ is absolutely continuous on $(a, 1]$ for every $a \in (0, 1)$. Is $f$ necessarily absolutely continuous on $[0, 1]$? If $f$ is also of bounded variation on $[0, 1]$, is $f$ absolutely continuous on $[0, 1]$? If not, give counterexamples.

6. A real-valued function $f$ is Lipschitz with constant $M$ if

$$ (\forall x, y \in \mathbb{R}) |f(x) - f(y)| \leq M|x - y|. $$

Prove that if $f$ is Lipschitz with constant $M$ if and only if $f$ is absolutely continuous and $|f'| \leq M$ a.e.

7. Suppose $F_n$ is a sequence of increasing non-negative right continuous functions on $[0, 1]$ such that $\sup_n F_n(1) < \infty$. Let $F \overset{\text{def}}{=} \sum_{n=1}^{\infty} F_n$ and suppose $F(1) < \infty$. Prove that

$$ F'(x) = \sum_{n=1}^{\infty} F'_n(x) $$

for almost every $x$.

8. Suppose $f$ is absolutely continuous on $[0, 1]$. Write $f(A) \overset{\text{def}}{=} \{f(x) : x \in A\}$ for $A \subset [0, 1]$. Prove that if $A$ has Lebesgue measure 0, then $f(A)$ has Lebesgue measure 0.

9. If $f$ is real-valued and differentiable at each point of $[0, 1]$, is $f$ necessarily absolutely continuous on $[0, 1]$? If not, find a counterexample.

10. Find an increasing function $f$ such that $f' = 0$ a.e. but $f$ is not constant on any open interval.

11. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and, for $y \in \mathbb{R}$, let $M(y)$ be the number of points $x \in [a, b]$ such that $f(x) = y$. $M(y)$ may be finite or infinite. Prove that $M$ is Borel measurable and $\int M(y) \, dy$ equals the total variation of $f$ on $[a, b]$.

12. Let $\alpha \in (0, 1)$. Find a Borel subset $E \subset [-1, 1]$ such that

$$ \lim_{r \to 0^+} \frac{m(E \cap [-r, r])}{2r} = \alpha. $$

13. Suppose that $f$ is a real-valued continuous function on $[a, b]$ and the derivative $D^+ f$ is non-negative on $[a, b]$. Prove that $f(b) \geq f(a)$. Is this true if it is instead assumed that $D_+ f$ is non-negative on $[a, b]$?
14. Let
\[ f(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{-xy^2} \frac{1}{1+y^2} dy. \]

a. Find the derivative of \( f \).

b. Find an ordinary differential equation that \( f \) solves.

c. Find the solution to this ordinary differential equation to determine an explicit value for \( f(x) \).

15. Let \((X, \mathcal{A}, \mu)\) be a measure space where \( \mu(X) > 0 \) and let \( f \) be a real-valued integrable function. Define
\[ g(x) \overset{\text{def}}{=} \int_X |f(y) - x| \mu(dy) \]
for \( x \in \mathbb{R} \).

a. Prove that \( g \) is absolutely continuous.

b. Prove that \( \lim_{x \to \infty} g(x) = \infty \) and \( \lim_{x \to -\infty} g(x) = \infty \).

c. Find \( g' \) and prove that \( g(x_0) = \inf_{x \in \mathbb{R}} g(x) \) if and only if
\[ \mu(\{ y : f(y) > x_0 \}) = \mu(\{ y : f(y) < x_0 \}). \]

16. Suppose \( A \subset [0,1] \) has Lebesgue measure zero. Find an increasing function \( f : [0,1] \to \mathbb{R} \) that is absolutely continuous, but
\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \infty \]
for each \( x \in A \).

17. Suppose that \( \mu \) is a measure on the Borel \( \sigma \)-algebra on\([0,1]\) and every real-valued and continuously differentiable function \( f \) satisfies
\[ \left| \int f'(x) \mu(dx) \right| \leq \left( \int_0^1 f(x)^2 \, dx \right)^{1/2}. \]

a. Show that \( \mu \) is absolutely continuous with respect to Lebesgue measure on \([0,1]\).

b. Let \( g \) be the Radon-Nikodym derivative of \( \mu \) with respect to Lebesgue measure. Prove that there exists a constant \( c > 0 \) such that
\[ (\forall x, y \in [0,1]) \ |g(x) - g(y)| \leq c|x - y|^{1/2}. \]
18. Fix $p > 1$ and suppose $f, g \in L^p(\mathbb{R})$. Define

$$H(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} |f(x) + tg(x)|^p \, dx$$

for $t \in \mathbb{R}$. Prove that $H$ is a differentiable function and find its derivative.

19. Show that $L^\infty$ is complete.

20. Suppose $1 \leq p \leq \infty$. Prove that the collection of simple functions is dense in $L^p$.

21. Suppose $p \geq 1$. Prove that

$$\int |f(x)|^p \, dx = \int_0^\infty pt^{p-1}m(\{x : |f(x)| \geq t\}) \, dt.$$  

(Hint: see HW 4, exercise 3, which is Exercise 11.3 on textbook p.88.)

22. Suppose $f$ is a measurable function on the measure space $(\mathbb{R}, \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel $\sigma$-algebra and $m$ is Lebesgue measure. Prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

23. When does equality hold in Hölder’s inequality?

24. When does equality hold in Minkowski’s inequality?

25. Provide a counterexample to the claim that $1 < p < q < \infty$ implies $L^p \subset L^q$.

26. Provide a counterexample to the claim that $1 < p < q < \infty$ implies $L^q \subset L^p$.

27. Define

$$g_n(x) \overset{\text{def}}{=} n\chi_{[0,n^{-1}]}(x).$$

a. Show that if $f \in L^2([0,1])$, then

$$\int_0^1 f(x)g_n(x) \, dx \to 0$$

as $n \to \infty$.

b. Show that there exists $f \in L^1([0,1])$ such that $\int_0^1 f(x)g_n(x) \, dx \not\to 0$.

28. Suppose $\mu$ is a finite measure on the Borel subsets of $\mathbb{R}$ such that

$$f(x) = \int_{\mathbb{R}} f(x + t) \mu(dt), \quad \text{a.e.},$$

whenever $f$ is real-valued, bounded, and integrable. Prove that $\mu(\{0\}) = 1$. 

4
29. Suppose $\mu$ is a measure with $\mu(X) = 1$ and $f \in L^r$ for some $r > 0$, where $L^r$ is defined for $0 < r < 1$ exactly as in equation 15.1 on textbook p.131. Prove that

$$\lim_{p \to 0} \|f\|_p = \exp \left( \int \log |f| \, d\mu \right),$$

where the convention $\exp(-\infty) = 0$ is used if needed.

30. Suppose $1 < p < \infty$ and $q$ is the conjugate exponent to $p$. Suppose $f_n \to f$ a.e. and $\sup_n \|f_n\|_p < \infty$. Prove that if $g \in L^q$, then

$$\lim_{n \to \infty} \int f_n g = \int fg.$$

31. Does the result in Exercise 30 hold in the case $p = 1$ with $q = \infty$? If not, give a counterexample.

32. Does the result in Exercise 30 hold in the case $p = \infty$ with $q = 1$? If not, give a counterexample.

33. If $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for some $p \in [1, \infty)$, prove that

$$\|f \ast g\|_p \leq \|f\|_1 \|g\|_p.$$

34. Suppose $p \in (1, \infty)$ and $q$ is its conjugate exponent. Prove that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $f \ast g$ is uniformly continuous and $f \ast g(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$.

35. Show that if $f$ and $g$ are continuous with compact support, then $f \ast g$ is continuous with compact support.

36. Suppose $f \in L^\infty$, $f_h(x) = f(x + h)$, and

$$\lim_{h \to 0} \|f_h - f\|_\infty = 0.$$ 

Prove that there exists a uniformly continuous function $g$ on $\mathbb{R}$ such that $f = g$ a.e.

37. Let $p \in [0, \infty)$ and suppose $\mu$ is a finite measure. Prove that $f \in L^p(\mu)$ if and only if

$$\sum_{n=1}^{\infty} (2^n)^p \mu(\{x : |f(x)| > 2^n\}) < \infty.$$
38. Suppose \( \mu(X) = 1 \) and \( f \) and \( g \) are non-negative functions such that \( fg \geq 1 \) a.e. Prove that \[
\left( \int f \, d\mu \right) \left( \int g \, d\mu \right) \geq 1.
\]

39. Suppose \( f : [0, \infty) \to \mathbb{R}, \) \( f(1) = 0, \) \( f' \) exists and is continuous and bounded, and \( f' \in L^2([1, \infty)) \). Let \( g(x) \) \( \overset{\text{def}}{=} f(x)/x \). Prove that \( g \in L^2([1, \infty)) \).

40. Find an example of a measurable \( f : [0, \infty) \to \mathbb{R} \) such that \( f(1) = 0, \) \( f' \) exists and is continuous and bounded, \( f' \in L^1([1, \infty)) \), but the function \( g(x) \) \( \overset{\text{def}}{=} f(x)/x \) is not in \( L^1([1, \infty)) \).

41. Prove the generalized Minkowski inequality: If \((X, A, \mu)\) and \((X, B, \nu)\) are measure spaces, \( f \) is measurable with respect to \( A \times B \), and \( 1 < p < \infty \), then
\[
\left( \int_X \left( \int_Y |f(x,y)| \, \nu(dy) \right)^p \, \mu(dx) \right)^{1/p} \leq \int_Y \left( \int_X |f(x,y)|^p \, \mu(dx) \right)^{1/p} \, \nu(dy).
\]
This could be rephrased as
\[
\left\| \|f\|_{L^1(\nu)} \right\|_{L^p(\mu)} \leq \left\| \|f\|_{L^p(\mu)} \right\|_{L^1(\nu)}.
\]
(Put \( Y = \{1, 2\} \), let \( g_1(x) \) \( \overset{\text{def}}{=} f(x, 1) \) and \( g_2 \) \( \overset{\text{def}}{=} f(x, 2) \), and take \( \nu(dy) = \delta_1(dy) + \delta_2(dy) \) where \( \delta_1, \delta_2 \) are point masses at 1, 2, respectively, to recover the usual Minkowski inequality, Proposition 15.3 on textbook p.133.)

42. Does the generalized Minkowski inequality in Exercise 41 extend to the case \( p = 1 \)? If not, give a counterexample.

43. Does the generalized Minkowski inequality in Exercise 41 extend to the case \( p = \infty \)? If not, give a counterexample.

44. Suppose \( \alpha \in (0, 1) \) and define \( K(x) \) \( \overset{\text{def}}{=} |x|^{-\alpha} \) for \( x \in \mathbb{R} \). Note that \( K \notin L^p(\mathbb{R}) \) for any \( p \geq 1 \). Prove that if \( f \) is non-negative, real-valued, and integrable on \( \mathbb{R} \) and \( \]
\[
g(x) \overset{\text{def}}{=} \int_{\mathbb{R}} f(x-t)K(t) \, dt,
\]
then \( g \) is finite a.e.

45. Suppose \( p > 1 \) and \( q \) is its conjugate exponent, \( f \) is an absolutely continuous function on \([0, 1]\) with \( f' \in L^p \), and \( f(0) = 0 \). Prove that if \( g \in L^q \), then
\[
\int_0^1 |fg| \, dx \leq \left( \frac{1}{p} \right)^{1/p} \|f'\|_p \|g\|_q.
\]
46. Suppose $f : \mathbb{R} \to \mathbb{R}$ is in $L^p$ for some $p > 1$ and also in $L^1$. Prove that there exist constants $c > 0$ and $\alpha \in (0, 1)$ such that
\[ \int_A |f(x)| \, dx \leq cm(A)^\alpha \]
for every Borel measurable set $A \subset \mathbb{R}$, where $m$ is Lebesgue measure.

47. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and there exist constants $c > 0$ and $\alpha \in (0, 1)$ such that
\[ \int_A |f(x)| \, dx \leq cm(A)^\alpha \]
for every Borel measurable set $A \subset \mathbb{R}$, where $m$ is Lebesgue measure. Prove that there exists $p > 1$ such that $f \in L^p$.

48. Suppose $1 < p < \infty$, $f : (0, \infty) \to \mathbb{R}$, and $f \in L^p$ with respect to Lebesgue measure. Define
\[ g(x) = \frac{1}{x} \int_0^x f(y) \, dy. \]
Prove that
\[ \|g\|_p \leq \frac{p}{p-1} \|f\|_p. \]
(This is known as Hardy’s inequality.)

49. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and suppose $K : X \times X \to \mathbb{R}$ is measurable with respect to $\mathcal{A} \times \mathcal{A}$. Suppose there exists $M < \infty$ such that
\[ \int_X |K(x, y)| \mu(dy) \leq M \]
for each $x$, and
\[ \int_X |K(x, y)| \mu(dx) \leq M \]
for each $y$. If $f$ is measurable and real-valued, define
\[ Tf(x) \overset{\text{def}}{=} \int_X K(x, y)f(y) \mu(dy) \]
if the integral exists.

a. Show that \( \|Tf\|_1 \leq M \|f\|_1 \).

b. If $1 < p < \infty$, show that \( \|Tf\|_p \leq M \|f\|_p \).

50. Suppose $A$ and $B$ are two Borel measurable subsets of $\mathbb{R}$, each with finite strictly positive Lebesgue measure. Show that $\chi_A \ast \chi_B$ is a continuous non-negative function that is not identically equal to zero.
51. Suppose $A$ and $B$ are two Borel measurable subsets of $\mathbb{R}$ with strictly positive Lebesgue measure. Show that

\[ C \overset{\text{def}}{=} \{ x + y : x \in A, y \in B \} \]

contains a non-empty open interval.

52. Suppose $1 < p < \infty$ and $q$ is the conjugate exponent of $p$. Prove that if $H$ is a bounded complex-valued linear functional on $L^p$, then there exists a complex-valued measurable function $g \in L^q$ such that $H(f) = \int fg$ for all $f \in L^p$ and $\|H\| = \|g\|_q$. 