Please do Exercises 2, 3, 5*, 6, 10*, 11, 12, 15, 16*, 17.
Exercises marked with (*) are especially important and you may wish to focus extra attention on those.
You are encouraged to try the other problems in this list as well.
Note: “textbook” refers to “Real Analysis for Graduate Students,” version 2.1, by Richard F. Bass. Some of these exercises originate from that source.

1. Let \( \langle f, g \rangle \overset{\text{def}}{=} \int_0^1 f(x)g(x)\,dx \) be the usual inner product in \( L^2([0, 1]) \). Prove that \( C([0, 1]) \) is not a Hilbert space with respect to this inner product and its derived norm.

2. Suppose that \( \{x_n\} \) is a sequence in a Hilbert space \( H \). Suppose \( \|x_n\| \to \|x\| \) and \( (\forall y \in H) \langle x_n, y \rangle \to \langle x, y \rangle \) as \( n \to \infty \). Prove that \( \|x_n - x\| \to 0 \) as \( n \to \infty \).

3. Suppose \( M \) is a subspace of a Hilbert space \( H \).
   a. Prove that if \( M \) is closed, then \( (M^\perp)^\perp = M \).
   b. Find a counterexample where \( M \) is not closed and \( (M^\perp)^\perp \neq M \).

4. Prove that if \( H \) is infinite-dimensional, namely there are linearly independent subsets \( \{x_1, \ldots, x_n\} \) for all \( n = 1, 2, \ldots \), then the closed unit ball in \( H \) is not compact.

5. Suppose \( \{a_n : n = 1, 2, \ldots\} \) is a sequence of real numbers such that
\[
\sum_{n=1}^{\infty} a_n b_n < \infty
\]
whenever \( \sum_{n=1}^{\infty} b_n^2 < \infty \). Prove that \( \sum_{n=1}^{\infty} a_n^2 < \infty \).

6. Say that \( x_n \to x \) weakly in a Hilbert space \( H \) if \( \langle x_n, y \rangle \to \langle x, y \rangle \) for every \( y \in H \). Prove that if \( x_n \) is a sequence in \( H \) bounded by \( \sup_n \|x_n\| \leq 1 \), then there is a subsequence \( n_j \) and an element \( x \in H \) with \( \|x\| \leq 1 \) such that \( x_{n_j} \to x \) weakly as \( j \to \infty \).
7. If \( A \) is a Lebesgue measurable subset of \([0, 2\pi]\), prove that
\[
\lim_{n \to \infty} \int_A e^{inx} \, dx = 0.
\]
(This is a special case of the Riemann-Lebesgue lemma.)

8. Suppose that \( \mu, \nu \) are finite measures on a measurable space \((X, \mathcal{A})\), with \( \nu \ll \mu \). Assume that \( \nu(A) \leq \mu(A) \) for all measurable \( A \).

For real-valued \( f \in L^2(X, \mu) \), define \( L(f) \overset{\text{def}}{=} \int_X f \, d\mu \).

a. Show that \( L \) is a bounded linear functional on \( L^2(X, \mu) \).

b. Show that there exists a real-valued measurable function \( g \in L^2(X, \mu) \) such that \( L(f) = \int_X fg \, d\mu \) for all \( f \in L^2(X, \mu) \). (Hint: use theorem 19.10 on textbook p.188.)

c. Show that \( g \) from part b is the Radon-Nikodym derivative \( d\nu/d\mu \).

Note: by exercise 13.6 on textbook p.105, the assumption \( \nu(A) \leq \mu(A) \) imposes no restriction as one may replace \( \mu \) with \( \mu + \nu \). Hence, this is an alternate proof of the Radon-Nikodym theorem as a corollary of theorem 19.10.

9. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is continuous and 1-periodic, namely \( f(x+1) = f(x) \) for all \( x \in \mathbb{R} \). Prove that if \( \gamma \in \mathbb{R} \) is irrational, then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(j\gamma) = \int_0^1 f(x) \, dx.
\]
(This is a special case of the Birkhoff ergodic theorem.)

10. Suppose that \( M \) is a closed subspace of a Hilbert space \( H \). Fix \( x \in H \) and define \( x + M \overset{\text{def}}{=} \{ x + y : y \in M \} \).

a. Prove that \( x + M \) is a closed convex subset of \( H \).

b. Let \( Qx \) be the (unique) point of \( x + M \) with smallest norm and let \( Px = x - Qx \). (\( P \) is called the orthogonal projection of \( x \) onto \( M \).) Prove that \( P \) and \( Q \) are surjective from \( H \) to \( M \) and \( M^\perp \), respectively.

c. Prove that \( P \) and \( Q \) are linear mappings.

d. Prove that if \( y \in M \) then \( Py = y \) and \( Qy = 0 \).

e. Prove that if \( z \in M^\perp \) then \( Pz = 0 \) and \( Qz = z \).

f. Prove that \( \|w\|^2 = \|Pw\|^2 + \|Qw\|^2 \) for any \( w \in H \).
11. Suppose \( \{e_n\} \) is a countable orthonormal basis for a Hilbert space \( H \) and \( \{f_n\} \) is a countable orthonormal set such that
\[
\sum_n \|e_n - f_n\|^2 < 1.
\]
Prove that \( \{f_n\} \) is a basis.

12. Suppose that \( \{e_n\} \) and \( \{f_n\} \) are countable orthonormal bases for a Hilbert space \( H \). Define a linear transformation \( T : H \rightarrow H \) by
\[
T(\sum_n c_n e_n) = \sum_n c_n f_n.
\]
\begin{enumerate}
  \item Prove that \( T \) is continuous and compute the operator norm of \( T \).
  \item Prove that \( \langle Tx, Ty \rangle = \langle x, y \rangle \) for all \( x, y \in H \).
\end{enumerate}

13. Let \( H, G \) be Hilbert spaces. Say that a linear function \( T : H \rightarrow G \) is an isometry if \( \langle Tx, Ty \rangle_G = \langle x, y \rangle_H \) for every \( x, y \in H \).
\begin{enumerate}
  \item Prove that an isometry \( T \) is continuous and compute the operator norm of \( T \).
  \item Prove that if \( T \) is an isometry then it is injective.
  \item Must an isometry be surjective? Supply a proof or a counterexample.
  \item Suppose \( T_1 \) and \( T_2 \) are isometries. Must \( T_1 + T_2 \) be an isometry?
\end{enumerate}

14. Let \( H \) be a Hilbert spaces. Say that a linear function \( T : H \rightarrow H \) is selfadjoint if \( \langle Tx, y \rangle = \langle x, Ty \rangle \) for every \( x \in H \).
\begin{enumerate}
  \item Prove that if \( T \) is selfadjoint, then \( \langle Tx, x \rangle \) is real-valued for all \( x \in H \).
  \item Say that selfadjoint \( T \) is positive definite if \( \langle Tx, x \rangle > 0 \) for all \( x \neq 0 \). Prove that such \( T \) must be injective.
  \item Give an example of a positive definite \( T \) that is not surjective.
\end{enumerate}

15. Let \( X, Y \) be Hilbert spaces. Suppose that \( T : X \rightarrow Y \) is a bounded linear function. Prove that there exists a unique bounded linear function \( T^* : Y \rightarrow X \) satisfying
\[
\langle \forall x \in X \rangle (\forall y \in Y) \ \langle Tx, y \rangle_Y = \langle x, T^* y \rangle_X.
\]
(Such \( T^* \) is called the adjoint of \( T \).)

16. Suppose \( X, Y \) are Hilbert spaces, and write \( T^* : Y \rightarrow X \) for the adjoint of bounded linear function \( T : X \rightarrow Y \) as in exercise 15.
\begin{enumerate}
  \item Prove that \( (T^*)^* = T \). Conclude that \( \|T\| = \|T^*\| \).
\end{enumerate}
b. Show that \((T_1 + cT_2)^* = T_1^* + cT_2^*\) for any \(T_1, T_2\) and \(c \in \mathbb{C}\).

c. Prove that \(T^*T : X \to X\) is selfadjoint and, if \(T\) is injective, also positive definite.

d. Suppose that \(T\) is an isometry (see exercise 13). Prove that \(T^*T : X \to X\) is the identity, and \(TT^* : Y \to Y\) is an orthogonal projection.

e. Suppose that \(T\) is a surjective isometry. Prove that \(TT^* : Y \to Y\) is the identity.

17. Suppose that \(H\) is a separable Hilbert space and write \(\ell^2\) for the Hilbert space of square-summable sequences in \(\mathbb{C}\). Prove that there is a bijective linear isometry \(T : \ell^2 \to H\). (Hint: see exercise 12.)

18. Define the Haar function \(h : \mathbb{R} \to \mathbb{R}\) by

\[
h \overset{\text{def}}{=} \chi_{[0,1/2)} - \chi_{[1/2,1)}; \quad h(t) \overset{\text{def}}{=} \begin{cases} 0, & \text{if } t < 0 \text{ or } t \geq 1; \\ 1, & \text{if } 0 \leq t < 1/2; \\ -1, & \text{if } 1/2 \leq t < 1. \end{cases}
\]

For integers \(j, k\), define

\[
h_{j,k}(t) \overset{\text{def}}{=} 2^{j/2}h(2^jt - k).
\]

Prove that

\[
H \overset{\text{def}}{=} \{h_{j,k} : j, k \in \mathbb{Z}\}
\]

is an orthonormal basis for \(L^2(\mathbb{R})\).

Note: \(H\) is called the Haar basis. It is defined by the mother function \(h_{0,0} = h\), and its elements satisfy

\[
\text{supp } h_{j,k} = \left[ \frac{k}{2^j}, \frac{k+1}{2^j} \right].
\]