1. Prove that if $X$ is a non-empty set and $\mathcal{T}_1, \mathcal{T}_2$ are two topologies on $X$, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is also a topology on $X$.

2. Prove that if $X$ is a topological space, $G$ is open in $X$, and $A$ is dense in $X$, then $G = G \cap A$.

3. Let $X = \mathbb{R}^2$ with the usual norm topology and write $x \sim y$ if and only if $x = Ax$ for some matrix

$$A = A(\theta) \overset{\text{def}}{=} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

determined by some $\theta \in \mathbb{R}$. (Such an $A$ is called a Givens matrix and corresponds to a rotation by $\theta$ radians about the origin.)

a. Show that $\sim$ is an equivalence relation.

b. Show that the quotient space is homeomorphic to $[0, \infty) \subset \mathbb{R}$ with the usual topology.

4. Prove that every metric space is first countable.

5. Let $X$ be an uncountable set of points and let $\mathcal{T}$ consist of all subsets of $A$ such that $A^c$ is finite and let $\mathcal{T}$ also contain the empty set. Prove that $(X, \mathcal{T})$ is a topological space that is not first countable.

6. Give an example of a metric space which is not second countable.

7. Prove that $A \subset X$ is dense if and only if $A \cap G \neq \emptyset$ for every nonempty open set $G$. 


8. Let $X = \prod_{\alpha \in I} X_\alpha$, where $I$ is a nonempty index set. Prove that a net $\langle x_\beta \rangle$ in $X$ converges to $x$ if and only if the net $\langle \pi_\alpha(x_\beta) \rangle$ converges to $\pi_\alpha(x)$ for every $\alpha \in I$.

9. Let $f : X \to Y$ be a function. Prove that $f$ is continuous if and only if whenever $x \in X$ and $f(x) \in G$ for an open set $G \subset Y$, then there exists an open set $H \subset X$ with $x \in H$ such that $f(H) \subset G$.

10. Let $X$ and $Y$ be topological spaces and fix $y_0 \in Y$. Prove that $X \times \{y_0\}$, with the relative topology derived from $X \times Y$, is homeomorphic to $X$.

11. Let $X,Y,Z$ be topological spaces, $f : X \to Y$, and $g : Y \to Z$. Prove that if $f$ and $g$ are continuous functions then so is $g \circ f : X \to Z$.

12. Suppose that $X,Y$ are topological spaces and $f : X \to Y$ is a function such that $f(x_n) \to f(x)$ in $Y$ whenever $x_n \to x$ in $X$. Is $f$ necessarily continuous? If not, give a counterexample.

13. Prove that $f : X \to Y$ is continuous if and only if the net $\langle f(x_\alpha) \rangle$ converges to $f(x)$ whenever the net $\langle x_\alpha \rangle$ converges to $x$.

14. Let $X$ be the collection of Lebesgue measurable functions on $[0,1]$ furnished with the topology of pointwise convergence. Say that $f \sim g$ for $f,g \in X$ if $f = g$ a.e. Describe the quotient topology.

15. A set $A$ has the Lindelöf property if every open cover of $A$ has a countable subcover. Prove that a metric space $X$ has the Lindelöf property if and only if $X$ is separable.

16. Find an example of a compact set that is not closed.

17. Show that the product topology on $[0,1]^{\mathbb{N}}$ and the topology generated by the metric

$$
\rho(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|, \quad x = (x_1,x_2,\ldots), y = (y_1,y_2,\ldots),
$$

of equation 20.3, section 20.8, textbook p.223, are the same.

18. Let $\{X_\alpha\}, \alpha \in I$, be a non-empty collection of topological spaces and let $X = \prod_{\alpha \in I} X_\alpha$. A sequence $\{x_n\}$ in $X$ converges pointwise to $x$ if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each $\alpha \in I$. Prove that $x_n$ converges to $x$ pointwise if and only if $x_n$ converges to $x$ with respect to the product topology of $X$.

19. Let $X$ be the space of real-valued bounded functions on $[0,1]$. $X$ can be identified with $\mathbb{R}^{[0,1]}$ furnished with the product topology. Let $E$ be the set of Borel measurable functions on $[0,1]$.

a. Show that $E$ is dense in $X$.

b. Let $N$ be a subset in $[0,1]$ that is not Borel measurable and put $f = \chi_N$. Prove that there does not exist a sequence in $E$ that converges to $f$, but that every neighborhood of $f$ contains points of $E$. 

2
20. Prove that if \( I \) is a nonempty countable set and each \( X_\alpha, \alpha \in I \), is second countable, then \( \prod_{\alpha \in I} X_\alpha \) is second countable.

21. If \( X \) is a metric space, define

\[
A^\delta \overset{\text{def}}{=} \{ x \in X : d(x, A) < \delta \},
\]

where \( d(x, A) \overset{\text{def}}{=} \inf \{ d(x, y) : y \in A \} \). For nonempty compact subsets \( E, F \) of \( X \), define

\[
d_H(E, F) \overset{\text{def}}{=} \inf \{ \delta > 0 : E \subset F^\delta \text{ and } F \subset E^\delta \}.
\]

a. Prove that \( d_H \) is a metric. (This is called the Hausdorff metric.)

b. Suppose \( X \) is compact. Is the set of nonempty closed subsets with metric \( d_H \) necessarily compact? Supply a proof or a counterexample.

22. Prove that if \( \{ x_n \} \) is a Cauchy sequence in a metric space \( X \) and a subsequence of \( \{ x_n \} \) converges to a point \( x \), then the full sequence converges to \( x \).

23. Prove that if \( X \) is a topological space, then \( C(X) \) is a complete metric space.

24. Prove that a sequence \( \{ x_n \} \) converges to a point \( x \) if and only if every subsequence \( \{ x_{n_j} \} \) has a further subsequence that converges to \( x \).

25. Let \( A \) be a subset of a metric space \( X \). Prove that if \( A \) is totally bounded then \( \bar{A} \) is also totally bounded.

26. Let \( X \) be a topological space in which each singleton \( \{ x \} \) is closed. Prove that \( X \) is a T_1 space.

27. Find two disjoint closed subsets \( E, F \) of \( \mathbb{R} \) such that \( \inf \{ |x - y| : x \in E, y \in F \} = 0 \).

28. Prove that every metric space is a normal space.

29. Prove that \( X \) is a Hausdorff space if and only if every net converges to at most one point.

30. Show that a closed subspace of a normal space is normal.

31. Prove that \([0, 1]^{[0, 1]}\) with the product topology is not metrizable.

32. Prove that if \( X \) is metrizable and \( I \) is countable and nonempty, then \( X^I \) is metrizable.

33. Let \( X \) be a locally compact Hausdorff space and \( X^* \) its one point compactification. A continuous function \( f : X \to \mathbb{R} \) is said to vanish at infinity if, given \( \epsilon > 0 \), there exists a compact set \( K \subset X \) such that \( |f(x)| < \epsilon \) for all \( x \in K^c \). Prove that \( f \) vanishes at infinity if and only if \( f \) is the restriction to \( X \) of a continuous function \( \tilde{f} : X^* \to \mathbb{R} \) with \( \tilde{f}(\infty) = 0 \).
34. Prove that the Alexandroff one-point compactification of $\mathbb{R}^n$ is homeomorphic to the unit $n$-sphere $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

35. A sequence $\{f_n\} \subset C(X)$ is said to converge uniformly on compact sets to a function $f \in C(X)$ if $f_n \to f$ in $C(K)$ for every compact subset $K \subset X$.

   a. Give an example of a sequence $\{f_n\} \subset C(\mathbb{R})$ that converges uniformly to 0 on compact sets but that does not converge uniformly to 0 on $\mathbb{R}$.

   b. Let $X$ be a $\sigma$-compact locally compact Hausdorff space, $M > 0$, and $\{f_n\}$ an equicontinuous sequence in $C(X)$ such that $|f_n(x)| \leq M$ for all $x \in X$ and all $n$. Prove that there exists a subsequence that converges uniformly on compact sets.

36. Show that $\mathbb{R}^N$ is not locally compact.

37. Show that $C([0,1])$ is not locally compact.

38. Prove that if $\{X_\alpha : \alpha \in I\}$ is a nonempty collection of Hausdorff spaces such that $\prod_{\alpha \in I} X_\alpha$ is locally compact, then each $X_\alpha$ is also locally compact.

39. A real-valued function $f$ on a subset $X \subset \mathbb{R}$ is Hölder continuous of order $\alpha$ if there exists $M$ such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for all $x, y \in X$. Suppose $0 < \alpha \leq 1$ and let $X = [0, 1]$. Prove that

$$\left\{ f \in C([0,1]) : \sup_{x \in [0,1]} |f(x)| \leq 1, \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq 1 \right\}$$

is compact in $C([0,1])$.

40. Let $K : [0,1]^2 \to \mathbb{R}$ be continuous and let $L$ be the set of Lebesgue measurable functions $f$ on $[0,1]$ such that $\|f\|_\infty \leq 1$. For $f \in L$, define

$$Tf(x) = \int_0^1 K(x,y)f(y)\,dy.$$ 

Prove that $\{Tf : f \in L\}$ is an equicontinuous family in $C([0,1])$.

41. Prove that if $X$ is a compact metric space, then $C(X)$ is separable.

42. Let $X = [0, \infty]$ be the one point compactification of $[0, \infty)$, the nonnegative reals with the usual metric. Let $A$ be the collection of all finite linear combinations

$$\sum_{j=1}^n a_j e^{-\lambda_j x},$$

where the $a_j$ are real and each $\lambda_j \geq 0$. 

4
a. Prove that $A$ is a dense subset of $C(X)$.

b. Prove that if $f_1$ and $f_2$ are two continuous integrable functions from $[0, \infty)$ to $\mathbb{R}$ that vanish at infinity and which have the same Laplace transform, that is,

$$\int_0^\infty e^{-\lambda x} f_1(x) \, dx = \int_0^\infty e^{-\lambda x} f_2(x) \, dx$$

for all $\lambda \geq 0$, then $f_1(x) = f_2(x)$ for all $x$.

43. Suppose $X$ and $Y$ are compact Hausdorff spaces. Let $A$ be the collection of real-valued functions in $C(X \times Y)$ of the form

$$\sum_{i=1}^n a_i g_i(x) h_i(y),$$

where $n \geq 1$, each $a_i \in \mathbb{R}$, each $g_i \in C(X)$, and each $h_i \in C(Y)$. Prove that $A$ is dense in $C(X \times Y)$.

44. Let $X$ be a compact Hausdorff space and suppose $A$ is an algebra of continuous functions that separates points. Prove that either $A$ is dense in $C(X)$ or else there exists a point $x \in X$ such that $A = \{f \in C(X) : f(x) = 0\}$.

45. Prove that if $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous functions such that

$$\int_0^1 f(x) x^n \, dx = \int_0^1 g(x) x^n \, dx$$

for $n = 0, 1, 2, \ldots$, then $f = g$.

46. Let $X$ be the closed unit disk in the complex plane. A polynomial in $z$ and $\bar{z}$ is a function of the form

$$P(z) = \sum_{j=0}^n \sum_{k=0}^n a_{jk} z^j \bar{z}^k,$$

where $a_{jk}$ are complex numbers. Prove that if $f$ is a function in $C(X, \mathbb{C})$, then $f$ can be uniformly approximated by polynomials in $z$ and $\bar{z}$.

47. Prove that if $B$ is a Banach space, then $B$ is connected.

48. Prove that if $A$ is a convex subset of a Banach space, then $A$ is connected.

49. A topological space $X$ is arcwise connected (or path connected) if for all $x, y \in X$ there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

   a. Prove that if $X$ is arcwise connected, then $X$ is connected.

   b. Find a counterexample to show that connected does not imply arcwise connected.
50. If $X$ is a topological space, a component of $X$ is a connected subset of $X$ that is not properly contained in any other connected subset of $X$. Prove that each $x \in X$ is contained in a unique component of $X$.

51. A topological space $X$ is totally disconnected if its components are all single points.

   a. Prove that the Cantor set with the relative topology derived from the real line is totally disconnected.

   b. Prove that if $\{X_\alpha : \alpha \in I\}$ is a nonempty collection of totally disconnected spaces, then $X \overset{\text{def}}{=} \prod_{\alpha \in I} X_\alpha$ is totally disconnected.

52. Prove that a topological space $X$ is connected if and only if for each pair $x, y \in X$ there is a connected subspace of $X$ containing both $x$ and $y$.

53. Let $X$ be a connected space. Suppose there exists a function $f : X \to \mathbb{R}$ that is continuous and non-constant. Prove that $X$ is uncountable.

54. Suppose $\{A_\alpha : \alpha \in I\}$ is a non-empty collection of connected subsets of a topological space $X$ with the property that $A_\alpha \cap A_\beta \neq \emptyset$ for each $\alpha, \beta \in I$. Prove that $\bigcup_{\alpha \in I} A_\alpha$ is connected.