Math 5052
Measure Theory and Functional Analysis II
Homework Assignment 10
Prof. Wickerhauser
Due Friday, April 1, 2016

Please do Exercises 4, 6, 8, 10*, 13, 14, 18*, 19, 20, 21*
Exercises marked with (*) are especially important and you may wish to focus extra attention on those.
You are encouraged to try the other problems in this list as well.
Note: “textbook” refers to “Real Analysis for Graduate Students,” version 2.1, by Richard F. Bass. Some
of these exercises originate from that source.

1. Prove that if $A$, $B$, and $C$ are bounded operators from a Hilbert space to itself, then
   a. $A(BC) = (AB)C$;
   b. $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$.

2. Prove that if $A$ is a bounded symmetric operator, then so is $A^n$ for each $n \geq 1$.

3. Suppose $H = \mathbb{C}^n$ and $Ax$ is multiplication of the vector $x \in H$ by an $n \times n$ matrix $M$. Prove that $A^*x$
is multiplication of $x$ by the conjugate transpose of $M$.

4. Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and $F : X \times X \to \mathbb{C}$ a jointly measurable function such that $F(y,x) = \overline{F(x,y)}$ and
   \[\int \int_{X \times X} |F(x,y)|^2 \, d\mu(x)d\mu(y) < \infty.\]
   (This is equation 25.4 on textbook p.353, corrected for complex-valued $F$.) Define $A : X \to X$ by
   \[A f(x) \overset{\text{def}}{=} \int_X F(x,y)f(y) \, d\mu(y).\]
   (This is equation 25.5 on textbook p.353.) Prove that $A$ is a bounded symmetric operator.

5. If $C_1$ and $C_2$ are subsets of a Hilbert space and their closures are compact, prove that the closure of $C_1 + C_2 \{x + y : x \in C_1, \ y \in C_2\}$
is also compact.
6. Prove that if $H$ is a Hilbert space, $K$ is a compact symmetric operator on $H$, and $Y$ is a closed subspace of $X$, then the map $K|_Y$ is compact.

7. Suppose $K$ is a bounded compact symmetric operator with non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$ and corresponding eigenvectors $z_1, \ldots, z_n$. Prove that for each $n$,
$$\lambda_n = \max_{x \perp z_1, \ldots, z_{n-1}} \frac{\langle Kx, x \rangle}{\|x\|^2}.$$  
(This is known as the Rayleigh principle.)

8. Let $K$ be a bounded compact symmetric operator and let $z_1, \ldots, z_n$ be eigenvectors with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $X$ be the linear subspace spanned by $\{z_1, \ldots, z_n\}$. Prove that if $y \in X$, then $\langle Ky, y \rangle \geq \lambda_n \langle y, y \rangle$.

9. Prove that the $n$th largest non-negative eigenvalue for a compact bounded symmetric operator $K$ satisfies
$$\lambda_n = \max \left\{ \min_{x \in S_n} \frac{\langle Kx, x \rangle}{\|x\|^2} : S_n \text{ is a linear subspace of dimension } n \right\}.$$  
(This is known as Fisher’s principle.)

10. Prove that the $n$th largest non-negative eigenvalue for a compact bounded symmetric operator $K$ satisfies
$$\lambda_n = \min \left\{ \max_{x \in S_{n-1}} \frac{\langle Kx, x \rangle}{\|x\|^2} : S_{n-1} \text{ is a linear subspace of dimension } n-1 \right\}.$$  
(This is known as Courant’s principle.)

11. Say that $A$ is a positive operator if $\langle Ax, x \rangle \geq 0$ for all $x$. (In the case of matrices, the term used is positive semidefinite.) Suppose $A$ and $B$ are compact symmetric operators and that $A - B$ is also a positive operator. Suppose $A$ and $B$ have eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots$ and $\beta_1 \geq \beta_2 \geq \cdots$, respectively, arranged in decreasing order. Prove that $\alpha_k \geq \beta_k$ for all $k$.

12. Let $A$ be a compact symmetric operator. Find necessary and sufficient conditions on a continuous function $f$ such that $f(A)$ is a compact symmetric operator.

13. Let $A$ be a bounded symmetric operator. For $z, w \in \sigma(A)^c$, put $R_z \overset{\text{def}}{=} (z-A)^{-1}$ and $R_w \overset{\text{def}}{=} (w-A)^{-1}$. Prove the resolvent identity:
$$R_w - R_z = (z - w)R_w R_z.$$  

14. Suppose $A$ is a bounded symmetric operator and $f$ is a continuous function on $\sigma(A)$. Let $P_n$ be polynomials which converge uniformly to $f$ on $\sigma(A)$. Suppose $\lambda \notin \mathbb{C}$ and suppose that there exists $\epsilon > 0$ such that $d(\lambda, \sigma(P_n(A))) \geq \epsilon$ for each $n$. Prove that $\lambda \notin \sigma(f(A))$. 

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15. Let $H$ be a separable Hilbert space and suppose that $K : H \to H$ is a compact symmetric linear operator. Prove that $K$ is a positive operator if and only if all the eigenvalues of $K$ are non-negative.

16. Let $A$ be a bounded symmetric operator, not necessarily compact. Prove that if $A = B^2$ for some bounded symmetric operator $B$, then $A$ is a positive operator.

17. Let $A$ be a bounded symmetric operator whose spectrum is contained in $[0, \infty)$. Prove that $A$ has a positive square root, namely that there exists a bounded positive symmetric operator $B$ such that $A = B^2$.

18. Let $A$ be a bounded symmetric operator, not necessarily compact. Prove that $A$ is a positive operator if and only if $\sigma(A) \subset [0, \infty)$.

19. Let $A$ be a bounded symmetric operator. Prove that $\mu_{x,x}$ is a real non-negative measure.

20. Suppose that $A$ is a bounded symmetric operator, $C_1, \ldots, C_n$ are disjoint Borel measurable subsets of $\mathbb{C}$, and $a_1, \ldots, a_n$ are complex numbers. Prove that

$$\left\| \sum_{k=1}^{n} a_k E(C_k) \right\| = \max_{1 \leq k \leq n} |a_k|.$$ 

21. Prove that if $A$ is a bounded symmetric operator and $f$ is a bounded Borel measurable function, then

$$\|f(A)x\|^2 = \int_{\sigma(A)} |f(z)|^2 \, d\mu_{x,x}(z).$$

22. Prove that if $A$ is a bounded symmetric operator and $\{s_n\}$ is a sequence of simple functions such that $s_n$ converges uniformly to a Borel measurable function $f$ on $\sigma(A)$, then $\|f(A) - s_n(A)\| \to 0$. 

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