Math 5052  
Measure Theory and Functional Analysis II  
Homework Assignment 12  
Prof. Wickerhauser  
Due Monday, May 2, 2016

Read Chapter 21 (Probability) in the textbook.
Please do Exercises 4, 9, 15*, 18, 22, 26, 35*, 38, 39, 40*
Exercises marked with (*) are especially important and you may wish to focus extra attention on those.
You are encouraged to try the other problems in this list as well.
Note: “textbook” refers to “Real Analysis for Graduate Students,” version 2.1, by Richard F. Bass. Some of these exercises originate from that source.

1. Show that if \( X \) has a continuous distribution function \( F_X \) and \( Y = F_X(X) \), then \( Y \) has a density \( f_Y(x) = \mathbf{1}_{[0,1]}(x) \).

2. Find an example of a probability space and three events \( A, B, \) and \( C \) such that \( \Pr(A \cap B \cap C) = \Pr(A) \Pr(B) \Pr(C) \), but \( A, B, \) and \( C \) are not independent events.

3. Suppose that
\[
\Pr(X \leq x, Y \leq y) = \Pr(X \leq x) \Pr(Y \leq y)
\]
for all \( x, y \in \mathbb{R} \). Prove that \( X \) and \( Y \) are independent random variables.

4. Find a sequence of events \( \{A_n\} \) such that
\[
\sum_{n=1}^{\infty} \Pr(A_n) = \infty
\]
but \( \Pr(A_n \text{ i.o.}) = 0 \).

5. A random vector \( X = (X_1, \ldots, X_n) \) has a joint density \( f_X \) if \( \Pr(X \in A) = \int_A f_X(x) \, dx \) for all Borel subsets \( A \) of \( \mathbb{R}^n \). Here the integral is with respect to \( n \) dimensional Lebesgue measure.

a. Prove that if the joint density of \( X \) factors into the product of densities of the \( X_j \), namely
\[
f_X(x) = \prod_{j=1}^{n} f_j(x_j),
\]
for almost every \( x = (x_1, \ldots, x_n) \), then the \( X_j \) are independent.
b. Prove that if $X$ has a joint density and the $X_j$ are independent, then each $X_j$ has a density and the joint density of $X$ factors into the product of the densities of the $X_j$.

6. Suppose $\{A_n\}$ is a sequence of events, not necessarily independent, such that $\sum_{n=1}^{\infty} \Pr(A_n) = \infty$. Suppose in addition that there exists a constant $c$ such that for each $N \geq 1$,

$$\sum_{i,j=1}^{N} \Pr(A_{i.o.}) \leq c \left( \sum_{i=1}^{N} \Pr(A_i) \right)^2.$$ 

Prove that $\Pr(A_n \text{ i.o.}) > 0$.

7. Suppose $X$ and $Y$ are independent, $\mathbb{E}|X|^p < \infty$ for some $p \in [1, \infty)$, $\mathbb{E}|Y| < \infty$, and $\mathbb{E}Y = 0$. Prove that

$$\mathbb{E}(|X + Y|^p) \geq \mathbb{E}|X|^p.$$ 

8. Suppose that $X_i$ are independent random variables such that $\text{Var}X_i/i \to 0$ as $i \to \infty$. Suppose also that $\mathbb{E}X_i \to a$. Prove that $S_n/n$ converges in probability to $a$, where $S_n = \sum_{i=1}^{n} X_i$. (It is not assumed that the $X_i$ are identically distributed.)

9. Suppose $\{X_i\}$ is a sequence of independent mean zero random variables, not necessarily identically distributed. Suppose that $\sup_i \mathbb{E}X_i^4 < \infty$.

   a. If $S_n \overset{\text{def}}{=} \sum_{i=1}^{n} X_i$, prove that there exists a constant $c$ such that $\mathbb{E}S_n^4 \leq cn^2$.
   
   b. Prove that $S_n/n \to 0$ a.s.

10. Suppose $\{X_i\}$ is an i.i.d. sequence such that $S_n/n$ converges a.s., where $S_n \overset{\text{def}}{=} \sum_{i=1}^{n} X_i$.

   a. Prove that $X_n/n \to 0$ a.s.
   
   b. Prove that $\sum_n \Pr(|X_n| > n) < \infty$.
   
   c. Prove that $\mathbb{E}|X_1| < \infty$.

11. Suppose $\{X_i\}$ is an i.i.d. sequence of random variables with $\mathbb{E}|X_1| < \infty$.

   a. Prove that the sequence $\{S_n/n\}$ is uniformly integrable by the definition in Exercise 7.16 on textbook p.58.
   
   b. Prove that $\mathbb{E}S_n/n$ converges to $\mathbb{E}X_1$.

12. Suppose $\{X_i\}$ is an i.i.d. sequence of random variables with $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}X_1 = 0$. Prove that

$$\frac{\max_{1 \leq k \leq n} |S_k|}{n} \to 0, \quad \text{a.s.}$$
13. Suppose that \( \{X_i\} \) is a sequence of independent random variables with mean zero such that \( \sum_i \text{Var} X_i < \infty \). Prove that \( S_n \) converges a.s. as \( n \to \infty \), where \( S_n \overset{\text{def}}{=} \sum_{i=1}^{n} X_i \).

14. Let \( \{X_i\} \) be a sequence of random variables. The tail \( \sigma \)-field is defined to be
\[
T = \bigcap_{n \geq 1} \sigma(X_n, X_{n+1}, \ldots).
\]
Let \( S_n = \sum_{i=1}^{n} X_i \).

a. Prove that the event \( (S_n \text{ converges}) \) is in \( T \).

b. Prove that the event \( (S_n/n > a) \) is in \( T \) for each real number \( a \).

15. Let \( \{X_i\} \) be a sequence of independent random variables and let \( T \) be the tail \( \sigma \)-field.

a. Prove that if \( A \in T \), then \( A \) is independent of \( \sigma(X_1, \ldots, X_n) \) for each \( n \).

b. Prove that if \( A \in T \), then \( A \) is independent of itself, and hence \( \Pr(A) \) is either 0 or 1.

Note: part b is known as the Kolmogorov 0-1 law.

16. Let \( \{X_i\} \) be an i.i.d. sequence of random variables. Prove that if \( \text{EX}_1^{+} = \infty \) and \( \text{EX}_1^{-} < \infty \), then \( S_n/n \to +\infty \) a.s., where \( S_n \overset{\text{def}}{=} \sum_{i=1}^{n} X_i \).

17. Let \( \mathcal{F} \subset \mathcal{G} \) be two \( \sigma \)-fields. Let \( H \) be the Hilbert space of \( \mathcal{G} \)-measurable random variables \( Y \) such that \( \text{E} Y^2 < \infty \) and let \( M \) be the subspace of \( H \) consisting of the \( \mathcal{F} \)-measurable random variables. Prove that if \( Y \in H \), then \( \text{E}[Y|\mathcal{F}] \) is equal to the orthogonal projection of \( Y \) onto the subspace \( M \).

18. Suppose \( \mathcal{F} \subset \mathcal{G} \) are two \( \sigma \)-fields and \( X \) and \( Y \) are bounded \( \mathcal{G} \)-measurable random variables. Prove that
\[
\text{E}[XE[Y|\mathcal{F}]] = \text{E}[YE[X|\mathcal{F}]].
\]

19. Let \( \mathcal{F} \subset \mathcal{G} \) be two \( \sigma \)-fields and let \( X \) be a bounded \( \mathcal{G} \)-measurable random variables. Prove that if
\[
\text{E}[XY] = \text{E}[XE[Y|\mathcal{F}]]
\]
for all bounded \( \mathcal{G} \)-measurable random variables \( Y \), then \( X \) is \( \mathcal{F} \)-measurable.

20. Suppose \( \mathcal{F} \subset \mathcal{G} \) are two \( \sigma \)-fields and that \( X \) is \( \mathcal{G} \)-measurable with \( \text{EX}^2 < \infty \). Set \( Y \overset{\text{def}}{=} \text{E}[X|\mathcal{F}] \). Prove that if \( \text{EX}^2 = \text{EY}^2 \), then \( X = Y \) a.s.

21. Suppose \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_N \) are \( \sigma \)-fields. Suppose \( A_i \) is a sequence of random variables adapted to \( \{\mathcal{F}_i\} \) such that \( A_1 \leq A_2 \leq \cdots \) and \( A_{i+1} - A_i \leq 1 \) a.s. for each \( i \). Prove that if \( \text{E}[A_N - A_i|\mathcal{F}] \leq 1 \) a.s. for each \( i \), then \( \text{E}A_N^2 < \infty \).
22. Let \( \{X_i\} \) be an i.i.d. sequence of random variables with \( \Pr(X_1 = 1) = \Pr(X_1 = -1) = \frac{1}{2} \). Let \( S_n \overset{\text{def}}{=} \sum_{i=1}^{n} X_i \). This sequence \( \{S_n\} \) is called a simple random walk. Let \( L \overset{\text{def}}{=} \max\{k \leq 9 : S_k = 1\} \wedge 9 \). 

Prove that \( L \) is not a stopping time with respect to the family of \( \sigma \)-fields \( \mathcal{F}_n = \sigma(S_1, \ldots, S_n) \).

23. Let \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \) be an increasing family of \( \sigma \)-fields and let \( \mathcal{F}_\infty \overset{\text{def}}{=} \sigma(\bigcup_n \mathcal{F}_n) \). If \( N \) is a stopping time, define
\[
\mathcal{F}_N \overset{\text{def}}{=} \{ A \in \mathcal{F}_\infty : A \cap (N \leq n) \in \mathcal{F}_n \text{ for all } n \}.
\]

a. Prove that \( \mathcal{F}_N \) is a \( \sigma \)-field.

b. If \( M \) is another stopping time with \( M \leq N \) a.s., and \( \mathcal{F}_M \) is defined analogously, prove that \( \mathcal{F}_M \subset \mathcal{F}_N \).

c. If \( X_n \) is a martingale with respect to \( \{\mathcal{F}_n\} \) and \( N \) is a stopping time bounded by the real number \( K \), prove that \( \mathbb{E}[X_n|\mathcal{F}_N] = X_N \).

24. Let \( \{X_i\} \) be a sequence of bounded i.i.d. random variables with mean 0. Let \( S_n = \sum_{i=1}^{n} X_i \).

a. Prove that there exists a constant \( c_1 \) such that \( M_n \overset{\text{def}}{=} e^{S_n - c_1 n} \) is a martingale.

b. Show that there exists a constant \( c_2 \) such that
\[
\Pr(\max_{1 \leq k \leq n} S_k > \lambda) \leq 2e^{-c_2 \lambda^2 / n}
\]
for all \( \lambda > 0 \).

25. Let \( \{X_i\} \) be a sequence of i.i.d. standard normal random variables with mean 0. Let \( S_n = \sum_{i=1}^{n} X_i \).

a. Prove that for each \( a > 0 \), \( M_n \overset{\text{def}}{=} e^{a S_n - a^2 n / 2} \) is a martingale.

b. Show that
\[
\Pr(\max_{1 \leq k \leq n} S_k > \lambda) \leq e^{-\lambda^2 / 2n}
\]
for all \( \lambda > 0 \).

26. Let \( \{X_n\} \) be a submartingale. Let
\[
A_n \overset{\text{def}}{=} \sum_{i=2}^{n} (X_i - \mathbb{E}[X_i|\mathcal{F}_{i-1}]).
\]

Prove that \( M_n \overset{\text{def}}{=} X_n - A_n \) is a martingale.

Note: this is known as the Doob decomposition of a submartingale.
27. Suppose $M_n$ is a martingale. Prove that if
\[
\sup_n EM_n^2 < \infty,
\]
then $M_n$ converges a.s. and also in $L^2$.

28. Set $(\Omega, \mathcal{F}, P)$ equal to $([0, 1], \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel $\sigma$-field on $[0, 1] \subset \mathbb{R}$ and $m$ is Lebesgue measure. Define
\[
X_n(\omega) \overset{\text{def}}{=} \begin{cases} 
1, & \text{if } \omega \in \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right) \text{ for some } k \leq 2^{n-1}; \\
-1, & \text{if } \omega \in \left[\frac{2k+1}{2^n}, \frac{2k+2}{2^n}\right) \text{ for some } k \leq 2^{n-1}.
\end{cases}
\]

a. Prove that $X_n$ converges weakly (in the probabilistic sense) to a nonzero random variable.
b. Prove that $X_n$ converges to 0 in the weak $L^2(m)$ sense, namely $E[X_nY] \to 0$ for all $Y \in L^2$.

29. Suppose $X_n$ is a sequence of random variables that converges weakly to a random variable $X$. Prove that the sequence $\{X_n\}$ is tight.

Note: see textbook p.275 for the definition of tight.

30. Suppose $X_n \to X$ weakly and $Y_n \to 0$ in probability. Prove that $X_nY_n \to 0$ in probability.

31. Given two probability measures $P$ and $Q$ on $[0, 1]$ with the Borel $\sigma$-field, define
\[
d(P, Q) \overset{\text{def}}{=} \sup \left\{ \left| \int f \, dP - \int f \, dQ \right| : f \in C^1, \|f\|_\infty \leq 1, \|f'\|_\infty \leq 1 \right\}.
\]

Here $C^1$ is the collection of continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$, and $f'$ is the derivative of $f$.

a. Prove that $d$ is a metric.
b. Prove that $P_n \to P$ weakly if and only if $d(P_n, P) \to 0$.

Note: This metric makes sense only for probabilities defined on $[0, 1]$. There are other metrics for weak convergence that work in more general situations.

32. Suppose $F_n \to F$ weakly and every point of $F$ is a continuity point. Prove that $F_n$ converges to $F$ uniformly over $x \in \mathbb{R}$:
\[
\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0.
\]

33. Suppose $X_n \to X$ weakly. Prove that $\phi_{X_n}$ converges uniformly to $\phi_X$ on each bounded interval.

34. Suppose $X_n$ is a collection of random variables that is tight. Prove that $\{\phi_{X_n}\}$ is equicontinuous on $\mathbb{R}$. 

5
35. Suppose \(X_n \rightarrow X\) weakly, \(Y_n \rightarrow Y\) weakly, and \(X_n\) and \(Y_n\) are independent for each \(n\). Prove that 
\(X_n + Y_n \rightarrow X + Y\) weakly.

36. \(X\) is said to be a \textit{gamma} random variable with parameters \(\lambda\) and \(t\) if \(X\) has density
\[
\frac{1}{\Gamma(t)} \lambda^t x^{t-1} e^{-\lambda x} 1_{(0, \infty)}(x),
\]
where \(\Gamma(t) \overset{\text{def}}{=} \int_0^\infty y^{t-1} e^{-y} dy\) is the Gamma function.

\textbf{a.} Prove that an exponential random variable with parameter \(\lambda\) is also a gamma random variable with parameters 1 and \(\lambda\).

\textbf{b.} Prove that if \(X\) is a standard normal random variable, then \(X^2\) is a gamma random variable with parameters 1/2 and 1/2.

\textbf{c.} Find the characteristic function of a gamma random variable.

\textbf{d.} Prove that if \(X\) is a gamma random variable with parameters \(t\) and \(\lambda\), and \(X\) and \(Y\) are independent, then \(X + Y\) is also a gamma random variable. Determine the parameters of \(X + Y\).

37. Suppose \(X_n\) is a sequence of independent random variables, not necessarily identically distributed, with \(\sup_n E|X_n|^3 < \infty\) and \(E X_n = 0\) and \(\text{Var} X_n = 1\) for each \(n\). Prove that \(S_n/\sqrt{n}\) converges weakly to a standard normal random variable, where \(S_n \overset{\text{def}}{=} \sum_{i=1}^n X_n\).

38. Suppose that \(X_n\) is a Poisson random variable with parameter \(n\) for each \(n\). Prove that \((X_n - n)/\sqrt{n}\) converges weakly to a standard normal random variable as \(n \rightarrow \infty\).

39. Suppose that \(P\) is a probability measure on the Borel subsets of \(\mathbb{R}^n\).

\textbf{a.} For \(\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n\), define \(X_k(\omega) \overset{\text{def}}{=} \omega_k\) for \(k = 1, \ldots, n\) and put \(X = (X_1, \ldots, X_n)\). Prove that the law \(P_X\) of \(X\) is equal to \(P\).

\textbf{b.} If \(P\) is a product measure, prove that the components of \(X\) are independent.

40. Prove that if \(X_t\) is a Brownian motion and \(a\) is a nonzero real number, then \(Y_t \overset{\text{def}}{=} aX_{a^2 t}\) is also a Brownian motion.

41. Let \(X_t\) be a Brownian motion. Fix \(n \geq 1\) and let \(M_k \overset{\text{def}}{=} X_{k/2^n}\).

\textbf{a.} Prove that \(M_k\) is a martingale.

\textbf{b.} Prove that if \(a \in \mathbb{R}\), then \(e^{aM_k - a^2(k/2^n)/2}\) is a martingale.

\textbf{c.} Prove that
\[
P\left(\sup_{t \leq r} X_t \geq \lambda\right) \leq e^{-\lambda^2/2r}.
\]
42. Let $X_t$ be a Brownian motion. Let
\[
A_n \overset{\text{def}}{=} \left( \sup_{t \leq 2^{n+1}} X_t > \sqrt{4 \cdot 2^n \log \log 2^n} \right).
\]

a. Prove that $\sum_{n=1}^{\infty} \Pr(A_n) < \infty$.
b. Prove that
\[
\lim sup_{t \to \infty} \frac{X}{\sqrt{t \log \log t}} < \infty, \quad \text{a.s.}
\]

Note: these results are part of what is called the law of the iterated logarithm for Brownian motion.

43. Let $X_t$ be a Brownian motion. Let $M > 0$, $t_0 > 0$, and
\[
B_n \overset{\text{def}}{=} \left( X_{t_0 + 2^{-n}} - X_{t_0 + 2^{-n-1}} > M 2^{-n-1} \right).
\]

a. Prove that $\sum_{n=1}^{\infty} \Pr(B_n) = \infty$.
b. Prove that the function $t \mapsto X_t(\omega)$ is not differentiable at $t = t_0$.
c. Prove that except for $\omega$ in a null set, the function $t \mapsto X_t(\omega)$ is not differentiable at almost every $t$ with respect to Lebesgue measure on $[0, \infty)$.

Note: it can be shown by a different proof that the function $t \mapsto X_t(\omega)$ is nowhere differentiable except for $\omega$ in a null set.

44. Let $X_t$ be a Brownian motion and let $h > 0$ be given. Prove that except for $\omega$ in a null set, there are times $t \in (0, h)$ for which $X_t(\omega) > 0$. (These times will depend on $\omega$.) Conclude that almost every Brownian path must oscillate quite a bit near 0.

45. Let $X_t$ be a Brownian motion on $[0, 1]$. Prove that $Y_t \overset{\text{def}}{=} X_1 - X_{1-t}$ is also a Brownian motion.