Independence: $GH^* = HG^* = 0 \in \mathbf{Mat}(M \times M)$. Thus the column space of G^* is orthogonal to the column space of H^*. As a consequence, for all $\mathbf{x} \in \mathbf{E}^{2M}$, we have $G^*G\mathbf{x} \perp H^*H\mathbf{x}$.

Completeness: $2H^*H + \frac{1}{2}G^*G = Id \in \mathbf{Mat}(2M \times 2M)$. Thus every $\mathbf{x} \in \mathbf{E}^{2M}$ may be written as $\mathbf{x} = \mathbf{s} + \mathbf{d}$, for some $\mathbf{s}$ in the M-dimensional column space of H^* and some $\mathbf{d}$ in the M-dimensional column space of G^*.

Running Averages. Fix $K > 1$ and for $0 \leq m \leq N - K$ and $0 \leq n < N$ let

$$A(m,n) = \begin{cases} \frac{1}{K}, & \text{if } m \leq n < m + K; \\ 0, & \text{otherwise.} \end{cases} \tag{2.65}$$

Then $A\mathbf{x}(m)$ is the average of the K components of $\mathbf{x}$ starting with $x(m)$. The associated matrix is

$$\begin{pmatrix} \frac{1}{K} & \frac{1}{K} & \dots & \frac{1}{K} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{K} & \dots & \frac{1}{K} & \frac{1}{K} & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{K} & \frac{1}{K} & \dots & \frac{1}{K} \end{pmatrix} = \frac{1}{K} \sum_{m \in M} \sum_{n=m}^{m+K-1} \mathbf{e}_{m,n}. \tag{2.66}$$

As usual, out-of-range coefficients are set to zero.

2.3 Exercises

1. How many vertices and edges are there in the 5-cube, the unit cube in Euclidean 5-space? Find a formula in terms of N for the number of vertices and edges of an N-cube.

2. Find an example subspace of $\mathbf{R}^N$ of dimension k for each $k = 0, 1, \dots, N$.

3. Show that the system of inequalities 2.14, 2.15, and 2.16 is sharp for every N by finding example vectors in $\mathbf{C}^N$ that give equality.

4. Prove that the following are equivalent:

   (i) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ and $\|a\mathbf{x}\| = |a|\,\|\mathbf{x}\|$, for all vectors $\mathbf{x}, \mathbf{y}$ and scalars a;

   (ii) $\|a\mathbf{x} + b\mathbf{y}\| \leq |a|\,\|\mathbf{x}\| + |b|\,\|\mathbf{y}\|$, for all vectors $\mathbf{x}, \mathbf{y}$ and scalars a, b.

5. Show that for any subset $\mathbf{Y}$ of an inner product space $\mathbf{X}$, we have $\mathbf{Y} \cap \mathbf{Y}^{\perp} = \{\mathbf{0}\}$, and $\mathbf{Y} \subset \left(\mathbf{Y}^{\perp}\right)^{\perp}$. (This is Lemma 2.5.)

6. Suppose that $\mathbf{Y} = \operatorname{span}\{\mathbf{y}_n : n = 1, \dots, N\}$. Show that if $\langle \mathbf{x}, \mathbf{y}_n \rangle = 0$ for all n, then $\mathbf{x} \in \mathbf{Y}^{\perp}$. (This is Lemma 2.6.)