Fraying may be performed over an arbitrary reach interval $[\alpha-\epsilon,\alpha+\epsilon]$, using the formula:

$$F(r,\alpha,\epsilon)u(t) = \begin{cases} r(\frac{t-\alpha}{\epsilon})u(t) + r(\frac{\alpha-t}{\epsilon})u(2\alpha-t), & \text{if } \alpha < t < \alpha+\epsilon, \\ \bar{r}(\frac{\alpha-t}{\epsilon})u(t) - \bar{r}(\frac{t-\alpha}{\epsilon})u(2\alpha-t), & \text{if } \alpha-\epsilon < t < \alpha, \\ u(t), & \text{otherwise.} \end{cases} \tag{3.20}$$

The formula for $S(r,\alpha,\epsilon)$, or splicing over $[\alpha-\epsilon,\alpha+\epsilon]$, is similar and left as an exercise. It is mostly shown in Equation 3.23 further on.

The boundary conditions at α will be the same as the boundary conditions at zero described in Lemma 3.4. Likewise, splicing over this reach interval undoes the boundary conditions at α. Every $\epsilon > 0$ will yield the same boundary conditions.

Suppose $F_1 = F(r_1,\alpha_1,\epsilon_1)$ and $F_2 = F(r_2,\alpha_2,\epsilon_2)$ are fraying operators with reach intervals $B_1 = [\alpha_1-\epsilon_1,\alpha_1+\epsilon_1]$ and $B_2 = [\alpha_2-\epsilon_2,\alpha_2+\epsilon_2]$, respectively. If B_1 and B_2 are disjoint, then F_1 and F_2 can be evaluated as follows:

$$F_1F_2u(t) = \begin{cases} F_1u(t), & \text{if } t \in B_1; \\ F_2u(t), & \text{if } t \in B_2; \\ u(t), & \text{otherwise.} \end{cases} \tag{3.21}$$

The same formula may be used to evaluate $F_2F_1u(t)$, so the operators F_1 and F_2 commute. Likewise, splicing operators $S_1 = S(r_1,\alpha_1,\epsilon_1)$ and $S_2 = S(r_2,\alpha_2,\epsilon_2)$ will commute with each other:

$$S_1S_2v(t) = \begin{cases} S_1v(t), & \text{if } t \in B_1; \\ S_2v(t), & \text{if } t \in B_2; \\ v(t), & \text{otherwise,} \end{cases} = S_2S_1v(t). \tag{3.22}$$

Similar formulas show that S_1 commutes with F_2 and S_2 commutes with F_1. The remaining pairs S_1, F_1 and S_2, F_2 commute because they are inverses.

Let $\alpha < \beta$ define an interval $I = [\alpha,\beta]$, and choose $0 < \epsilon < \frac{1}{2}(\beta-\alpha)$. A smooth function u frayed at $t=\alpha$ and $t=\beta$ with reach intervals $B_\epsilon(\alpha)$ and $B_\epsilon(\beta)$, respectively, may have its ends spliced together with the *loop* operator:

$$\begin{aligned} L(r,[\alpha,\beta],\epsilon)u(t) &= \begin{cases} \bar{r}(\frac{t-\alpha}{\epsilon})u(t) - r(\frac{\alpha-t}{\epsilon})u(\alpha+\beta-t), & \text{if } \alpha < t \le \alpha+\epsilon, \\ r(\frac{\beta-t}{\epsilon})u(t) + \bar{r}(\frac{t-\beta}{\epsilon})u(\alpha+\beta-t), & \text{if } \beta-\epsilon \le t < \beta, \\ u(t), & \text{otherwise;} \end{cases} \\ &= \begin{cases} S(r,\alpha,\epsilon)u_I(t), & \text{if } \alpha < t \le \alpha+\epsilon, \\ S(r,\beta,\epsilon)u_I(t), & \text{if } \beta-\epsilon \le t < \beta, \\ u(t), & \text{otherwise.} \end{cases} \end{aligned} \tag{3.23}$$

Here u_I is the periodic extension of u from its localization to $I = [\alpha,\beta]$, as defined in Equation 3.7.

The *smooth local periodization* of a function is a combination of fraying at two points and splicing into a loop. Namely, suppose $u = u(t)$ is a smooth function and $I = [\alpha,\beta]$ is an interval. Choose a smooth rising cut-off function r and a positive