

Finally, observe that

$$r\left(\frac{t-\alpha}{\epsilon}\right) = \begin{cases} 1, & \text{if } t \geq \alpha + \epsilon, \\ 0, & \text{if } t \leq \alpha - \epsilon, \end{cases} \quad r\left(\frac{\alpha-t}{\epsilon}\right) = \begin{cases} 0, & \text{if } t \geq \alpha + \epsilon, \\ 1, & \text{if } t \leq \alpha - \epsilon. \end{cases}$$

These valuations result in the formula in Equation 3.20.

d. Performing the previous calculation, but starting from Equation 3.15, we get

$$\begin{aligned} S(r, \alpha, \epsilon)u(t) &= \tau_\alpha \delta_\epsilon S \delta_\epsilon^{-1} \tau_\alpha^{-1} u(t) \\ &= \begin{cases} \bar{r}\left(\frac{t-\alpha}{\epsilon}\right)u(t) - r\left(\frac{\alpha-t}{\epsilon}\right)u(2\alpha-t), & \text{if } \alpha < t \leq \alpha + \epsilon, \\ r\left(\frac{\alpha-t}{\epsilon}\right)u(t) + \bar{r}\left(\frac{t-\alpha}{\epsilon}\right)u(2\alpha-t), & \text{if } \alpha - \epsilon \leq t < \alpha, \\ u(t), & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, since the inverse of $\tau_\alpha \delta_\epsilon$ is $\delta_\epsilon^{-1} \tau_\alpha^{-1} = \delta_{1/\epsilon} \tau_{-\alpha}$, we can simplify

$$\begin{aligned} S(r, \alpha, \epsilon)F(r, \alpha, \epsilon) &= (\tau_\alpha \delta_\epsilon S \delta_\epsilon^{-1} \tau_\alpha^{-1}) (\tau_\alpha \delta_\epsilon F \delta_\epsilon^{-1} \tau_\alpha^{-1}) \\ &= \tau_\alpha \delta_\epsilon S F \delta_\epsilon^{-1} \tau_\alpha^{-1} = \tau_\alpha \delta_\epsilon \delta_\epsilon^{-1} \tau_\alpha^{-1} = Id, \end{aligned}$$

since $SF = Id$. The other order, $F(r, \alpha, \epsilon)S(r, \alpha, \epsilon) = Id$, likewise follows from $FS = Id$. \square

3. **Solution:** Suppose that r is a rising cut-off function and $\epsilon < T/2$. Write $F_0 = F(r, 0, \epsilon)$ and $F_T = F(r, T, \epsilon)$ for the fraying operators at 0 and T , respectively. Since $u(t+T) = u(t)$ for all t , the function $v \stackrel{\text{def}}{=} F_0 F_T u$ satisfies $v(t+T) = v(t)$ for all $-\frac{T}{2} \leq t \leq \frac{T}{2}$. In particular, that means $v_I(t) = v(t)$ for all $-\epsilon \leq t \leq T + \epsilon$. Also, $v(t) = u(t)$ if t lies between ϵ and $T - \epsilon$, outside the reach intervals of F_0 and F_T .

Now put $I = [0, T]$. The second formula for the loop operator, Equation 3.23, simplifies to

$$L_I v(t) = \begin{cases} S(r, 0, \epsilon)v(t), & \text{if } 0 < t \leq \epsilon; \\ S(r, T, \epsilon)v(t), & \text{if } T - \epsilon \leq t < T; \\ v(t), & \text{otherwise.} \end{cases}$$

Writing $S_0 = S(r, 0, \epsilon)$ and $S_T = S(r, T, \epsilon)$ for the splicing operators at 0 and T , one calculates

$$L_I v(t) = \begin{cases} S_0 F_0 u(t) = u(t), & \text{if } 0 < t \leq \epsilon, \\ S_T F_T u(t) = u(t), & \text{if } T - \epsilon \leq t < T, \\ v(t) = u(t), & \text{if } \epsilon \leq t \leq T - \epsilon. \end{cases}$$

Thus $L_I F_0 F_T u = u$ on I , so $(L_I F_0 F_T u)_I = u_I = u$ on \mathbf{R} . \square

4. **Solution:** (i) $t > 1$ implies $\tilde{F}u(t) = \bar{1}u(t) = u(t)$ and $\tilde{S}u(t) = 1u(t) = u(t)$. Likewise, $t < -1$ implies $\tilde{F}u(t) = 1u(t) = u(t)$ and $\tilde{S}u(t) = \bar{1}u(t) = u(t)$.