imprecision. Conversely, a non-ideal instrument can have zero imprecision. An instrument that counts pennies can make only nonnegative integer measurements, and has quantization error $\frac{1}{2}$, but can have zero imprecision if it never miscounts since the ideal value is also a nonnegative integer.

Given a measurement $x$ of an ideal value $y$, we may hope that $x$ is close to $y$. Since our knowledge of $y$ is limited, we may at best label, or *calibrate*, the measuring instrument so that $x = E(Y_x)$, the mean value of the random variable $Y_x$ determined by the measurement $x$.

But even if the instrument is not calibrated, we may define the *inaccuracy* of the measurement as the root-mean-square error $\sqrt{E(|Y_x - x|^2)}$. It can be shown, using the Cauchy–Schwarz inequality of Lemma 2.4, that this is minimized when $x = E(Y_x)$, namely by calibration. Inaccuracy is never smaller than imprecision.

### Measurement density functions

Inaccuracy and imprecision may be computed from the *measurement density function*, which is a nonnegative function $f = f(x, y)$ giving the likelihood of any combinations of ideal value $y$ and measured value $x$:

$$\Pr(X \in [a_X, b_X], Y \in [a_Y, b_Y]) \stackrel{\text{def}}{=} \int_{a_X}^{b_X} \int_{a_Y}^{b_Y} f(x, y)\, dxdy. \tag{4.28}$$

Since $\Pr(X \in \mathbf{R}, Y \in \mathbf{R}) = 1$, we must have that $\iint_{R^2} f(x, y)\, dxdy = 1$. Such an $f$ is called a *joint probability density function* for the random variables $X$ and $Y$, giving likelihoods that they fall within particular ranges. With another normalization, it can be used to compute the likelihood that, given a measurement $x$, the random variable $Y = Y_x$ representing the ideal value falls in a particular range $[a_Y, b_Y]$:

$$\Pr(Y \in [a_Y, b_Y] \,|\, x) \stackrel{\text{def}}{=} \int_{a_Y}^{b_Y} f(y \,|\, x)\, dy, \tag{4.29}$$

where

$$f(y \,|\, x) \stackrel{\text{def}}{=} \frac{1}{c_x} f(x, y); \qquad c_x \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x, y)\, dy. \tag{4.30}$$

The normalizing constant $c_x$ must be finite and positive at each representable $x$; it guarantees that $\Pr(Y \in \mathbf{R} \,|\, x) = 1$. Such a normalized $f(y \,|\, x)$ is called a *conditional probability density function*. If in addition the instrument has finite imprecision, then the variance of $Y$ given $x$ will be finite for each representable $x$. This variance is comparable to $\int_{-\infty}^{\infty} y^2 f(y \,|\, x)\, dy$, which in turn is just a multiple of $\int_{-\infty}^{\infty} y^2 f(x, y)\, dy$. If this last integral is finite, we will say that the instrument is *focused*.

Recall that an instrument with measurement density $f$ is *calibrated* if, for each measurement $x$,

$$E(Y_x) = E(Y \,|\, x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} y f(y \,|\, x)\, dy = x.$$