2.1. VECTOR SPACES

and a function \( f \) belongs to \( L^2(\mathbb{R}) \) if and only if \( ||f|| \) is finite. Thus the nonzero constant functions, and more generally the nonzero polynomials, are not members of \( L^2(\mathbb{R}) \). Since \( ||f||^2 \) is called the energy of a function, \( L^2 \) is sometimes called the space of finite-energy signals.

\( L^2 \), like \textbf{Lip}, is infinite dimensional. Since there is no continuity assumption, we may build a simple set of basis functions from the indicator function of the unit interval \([0, 1]\):

\[
1(t) \overset{\text{def}}{=} \begin{cases} 
1, & \text{if } 0 \leq t < 1; \\
0, & \text{if } t < 0 \text{ or } t \geq 1.
\end{cases}
\]  

(2.25)

Given any integer \( k \), put \( e_k(t) \overset{\text{def}}{=} 1(t - k) \) to get the characteristic function of \([k, k + 1]\). The functions \( \{e_k : k \in \mathbb{Z}\} \) are clearly linearly independent in \( L^2(\mathbb{R}) \), and there are infinitely many of them. We can also introduce a scale index \( j \) and put \( e_{j,k}(t) \overset{\text{def}}{=} 2^{-j/2} 1(2^{-j}t - k) \), which is normalized to guarantee \( ||e_{j,k}|| = 1 \). The set \( \{e_{j,k} : j, k \in \mathbb{Z}\} \) is dense in \( L^2 \), but it is clearly not linearly independent. However, the fixed-scale functions \( E_j = \{e_{j,k} : k \in \mathbb{Z}\} \) are linearly independent, and given a function \( f \in L^2(\mathbb{R}) \) and \( \epsilon > 0 \), we can find a scale \( J \) and a function \( f_J \in \text{span } E_J \subset L^2(\mathbb{R}) \) satisfying \( ||f - f_J|| < \epsilon \).

2.1.3 Inner product spaces

An \textit{inner product space} \( X \) is a special kind of vector space in which there is also an \textit{inner product}. This is a scalar-valued function on pairs of vectors \( u, v \in X \), denoted by \( \langle u, v \rangle \), that must satisfy the following:

**Inner Product Axioms**

- **Hermitean symmetry**: For any \( u, v \in X \), \( \langle u, v \rangle = \overline{\langle v, u \rangle} \).
- **Positive definiteness**: If \( u \in X \) and \( u \neq 0 \), then \( \langle u, u \rangle > 0 \).
- **Linearity**: For any \( u, v, w \in X \) and any scalars \( c, d \), \( \langle cu + dv, w \rangle = c \langle u, v \rangle + d \langle u, w \rangle \).

Hermitean symmetry implies that \( \langle u, u \rangle \) is purely real. If all coordinates and scalars are real numbers, it reduces to the ordinary symmetry condition \( \langle u, v \rangle = \langle v, u \rangle \).

Positive definiteness implies \textit{nondegeneracy of the inner product}: \( \langle u, v \rangle = 0 \) for all \( v \in X \) only if \( u = 0 \). It also allows us to define a nondegenerate derived norm by the formula \( ||u|| \overset{\text{df}}{=} \sqrt{\langle u, u \rangle} \geq 0 \), just as in Euclidean \( N \)-space. Linearity implies that \( ||0||^2 = \langle 0, 0 \rangle = 0 \), so we have \( ||u|| = 0 \) if and only if \( u = 0 \).

By linearity and Hermitean symmetry, \( \langle cv + dw, u \rangle = c \langle v, u \rangle + d \langle w, u \rangle \). Thus \( \langle cu, cu \rangle = ||c||^2 ||u||^2 \), so the derived norm satisfies \( ||cu|| = ||c|| ||u|| \). We will see in Lemma 2.4 that the other sublinearity condition also holds, so a derived norm indeed satisfies the norm axioms.

If all scalars and coordinates are real numbers, the inner product is real-valued and linear in the first factor as well: \( \langle cv + dw, u \rangle = c \langle v, u \rangle + d \langle w, u \rangle \).
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