Orthonormal bases

In any inner product space, vectors $\mathbf{u}, \mathbf{v}$ are said to be *orthogonal* if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$. The zero vector is orthogonal to all other vectors, and no vector except $\mathbf{0}$ is orthogonal to itself.

If $\mathbf{Y}$ is any subset of an inner product space $\mathbf{X}$, then its *orthogonal complement* in $\mathbf{X}$ is a subspace, denoted $\mathbf{Y}^\perp$ and defined as follows:

$$\mathbf{Y}^\perp \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{y}\rangle = 0 \text{ for all } \mathbf{y} \in \mathbf{Y}.\} \tag{2.30}$$

For example, $\mathbf{X}^\perp = \{\mathbf{0}\}$ and $\{\mathbf{0}\}^\perp = \mathbf{X}$. Also, if $1 \leq m < N$ and we let $\mathbf{Y} = \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_m\} \subset \mathbf{E}^N$, then $\mathbf{Y}^\perp = \operatorname{span}\{\mathbf{e}_{m+1}, \ldots, \mathbf{e}_N\}$. It is left as an exercise to prove the following facts:

Lemma 2.5 *Suppose* $\mathbf{Y}$ *is a subset of an inner product space. Then* $\mathbf{Y} \cap \mathbf{Y}^\perp \subset \{\mathbf{0}\}$, *and* $\mathbf{Y} \subset \left(\mathbf{Y}^\perp\right)^\perp$. □

Lemma 2.6 *Suppose that* $\mathbf{Y} = \operatorname{span}\{\mathbf{y}_n : n = 1, \ldots, N\}$. *If* $\langle \mathbf{x}, \mathbf{y}_n\rangle = 0$ *for all* n, *then* $\mathbf{x} \in \mathbf{Y}^\perp$. □

A basis $\{\mathbf{b}_n\}$ in an inner product space is called an *orthogonal basis* if the vectors are pairwise orthogonal, that is, if $\langle \mathbf{b}_i, \mathbf{b}_j\rangle = 0$ when $i \neq j$, and an *orthonormal basis* if the vectors are pairwise orthogonal and also have unit length: $\langle \mathbf{b}_i, \mathbf{b}_i\rangle = 1$ for all i. These two conditions are summed up with the *Kronecker symbol* $\delta(i-j)$, defined by

$$\{\mathbf{b}_n\} \text{ is orthonormal} \iff \langle \mathbf{b}_i, \mathbf{b}_j\rangle = \delta(i-j) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{2.31}$$

The vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ defined previously form an orthonormal basis for $\mathbf{E}^N$. Any subset of an orthogonal or orthonormal basis inherits orthogonality and orthonormality, respectively.

Linearly independent vectors can be orthonormalized:

Theorem 2.7 (Gram-Schmidt) *Suppose* $\mathbf{X}$ *is an inner product space and* $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_N\} \subset \mathbf{X}$ *is a set of* N *linearly independent vectors. Then there is an orthonormal set* $\mathbf{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\} \subset \mathbf{X}$ *with* $\operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_k\} = \operatorname{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ *for all* $k = 1, \ldots, N$. *In particular,* $\operatorname{span}\mathbf{A} = \operatorname{span}\mathbf{B}$.

Proof: First note that the dimension of $\mathbf{X}$ is at least N, and that $\mathbf{b}_1 \neq \mathbf{0}$. We construct $\mathbf{a}_k$ from $\mathbf{b}_k$ inductively, starting with $\mathbf{a}_1 \stackrel{\text{def}}{=} \frac{1}{\|\mathbf{b}_1\|}\mathbf{b}_1$. Then $\|\mathbf{a}_1\| = 1$, $\operatorname{span}\{\mathbf{a}_1\} = \operatorname{span}\{\mathbf{b}_1\}$ since the two vectors are proportional, and the single-vector $\{\mathbf{a}_1\}$ is an orthogonal set. Now suppose that we have constructed an orthonormal set $\mathbf{A}_k \stackrel{\text{def}}{=} \{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ with the same span as $\mathbf{B}_k \stackrel{\text{def}}{=} \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$. Then the vector

$$\mathbf{a}'_{k+1} \stackrel{\text{def}}{=} \mathbf{b}_{k+1} - \sum_{i=1}^{k} \langle \mathbf{a}_i, \mathbf{b}_{k+1}\rangle\, \mathbf{a}_i$$

Independence: $GH^* = HG^* = 0 \in \mathbf{Mat}(M \times M)$. Thus the column space of G^* is orthogonal to the column space of H^*. As a consequence, for all $\mathbf{x} \in \mathbf{E}^{2M}$, we have $G^*G\mathbf{x} \perp H^*H\mathbf{x}$.

Completeness: $2H^*H + \frac{1}{2}G^*G = Id \in \mathbf{Mat}(2M \times 2M)$. Thus every $\mathbf{x} \in \mathbf{E}^{2M}$ may be written as $\mathbf{x} = \mathbf{s} + \mathbf{d}$, for some $\mathbf{s}$ in the M-dimensional column space of H^* and some $\mathbf{d}$ in the M-dimensional column space of G^*.

Running Averages. Fix $K > 1$ and for $0 \le m \le N - K$ and $0 \le n < N$ let

$$A(m,n) = \begin{cases} \frac{1}{K}, & \text{if } m \le n < m+K; \\ 0, & \text{otherwise.} \end{cases} \tag{2.65}$$

Then $A\mathbf{x}(m)$ is the average of the K components of $\mathbf{x}$ starting with $x(m)$. The associated matrix is

$$\begin{pmatrix} \frac{1}{K} & \frac{1}{K} & \dots & \frac{1}{K} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{K} & \dots & \frac{1}{K} & \frac{1}{K} & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{K} & \frac{1}{K} & \dots & \frac{1}{K} \end{pmatrix} = \frac{1}{K} \sum_{m \in M} \sum_{n=m}^{m+K-1} \mathbf{e}_{m,n}. \tag{2.66}$$

As usual, out-of-range coefficients are set to zero.

2.3 Exercises

1. How many vertices and edges are there in the 5-cube, the unit cube in Euclidean 5-space? Find a formula in terms of N for the number of vertices and edges of an N-cube.

2. Find an example subspace of $\mathbf{R}^N$ of dimension k for each $k = 0, 1, \dots, N$.

3. Show that the system of inequalities 2.14, 2.15, and 2.16 is sharp for every N by finding example vectors in $\mathbf{C}^N$ that give equality.

4. Prove that the following are equivalent:

   (i) $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ and $\|a\mathbf{x}\| = |a|\,\|\mathbf{x}\|$, for all vectors $\mathbf{x}, \mathbf{y}$ and scalars a;

   (ii) $\|a\mathbf{x} + b\mathbf{y}\| \le |a|\,\|\mathbf{x}\| + |b|\,\|\mathbf{y}\|$, for all vectors $\mathbf{x}, \mathbf{y}$ and scalars a, b.

5. Show that for any subset $\mathbf{Y}$ of an inner product space $\mathbf{X}$, we have $\mathbf{Y} \cap \mathbf{Y}^\perp \subset \{\mathbf{0}\}$, and $\mathbf{Y} \subset \left(\mathbf{Y}^\perp\right)^\perp$. (This is Lemma 2.5.)

6. Suppose that $\mathbf{Y} = \operatorname{span}\{\mathbf{y}_n : n = 1, \dots, N\}$. Show that if $\langle \mathbf{x}, \mathbf{y}_n \rangle = 0$ for all n, then $\mathbf{x} \in \mathbf{Y}^\perp$. (This is Lemma 2.6.)

A.2 ... to Chapter 2 Exercises

1. **Solution:** The 5-cube is created by sweeping the 4-cube through a fifth axis. It has $2^5 = 32$ vertices. We get 64 edges from the front and back 4-cubes, plus 16 new edges joining corresponding vertices on the front and back, for a total of 80.

   A general formula may be developed recursively. Let $v(N), e(N)$ be the number of vertices and edges, respectively, of an N-cube. We may consider the 0-cube to be a single point, so $v(0) = 1$ and $e(0) = 0$. The 1-cube is a unit line segment, so $v(1) = 2$ and $e(1) = 1$. Counting the vertices gives $v(N) = 2^N$, and for the edges we reason as in the previous paragraph to get the relation

   $$e(N+1) = 2e(N) + v(N) = 2e(N) + 2^N, \qquad N \geq 1.$$

   This has the closed form solution $e(N) = N2^{N-1}$, as may be proved by induction. □

2. **Solution:** The zero subspace $\{0\} \subset \mathbf{R}^N$ is zero-dimensional. For $k = 1, \ldots, N$, let $Y_k = \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$. Then Y_k is a k-dimensional subspace of $\mathbf{R}^N$. □

3. **Solution:** Fix N and let $\{\mathbf{e}_n\}$ be the standard basis for $\mathbf{C}^N$. Then $\mathbf{e}_1 = (1, 0, \ldots, 0)$ satisfies $\|\mathbf{e}_1\|_1 = \|\mathbf{e}_1\|_2 = \|\mathbf{e}_1\|_\infty = 1$, making the right column into equalities, while $\mathbf{f}_N = \mathbf{e}_1 + \cdots + \mathbf{e}_N = (1, 1 \ldots, 1)$ satisfies $\|\mathbf{f}_N\|_1 = N$, $\|\mathbf{f}_N\|_2 = \sqrt{N}$, and $\|\mathbf{f}_N\|_\infty = 1$, giving equalities in the left column. □

4. **Solution:** (i) $\Rightarrow$ (ii): Given vectors $\mathbf{x}', \mathbf{y}'$ and scalars a, b, substitute $\mathbf{x} \leftarrow a\mathbf{x}'$ and $\mathbf{y} \leftarrow b\mathbf{y}'$ into the triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ from (i), and then expand.

   (ii) $\Rightarrow$ (i): For the triangle inequality, take $a = b = 1$ in (ii). To show that $\|a\mathbf{x}\| = |a| \, \|\mathbf{x}\|$, start by taking $b = 0$ in condition (ii) to get $\|a\mathbf{x}\| \leq |a| \, \|\mathbf{x}\|$ for every scalar a. Then using this inequality with $a = 1 = \frac{1}{c} \cdot c$ shows that for all $c \neq 0$,

   $$\|\mathbf{x}\| = \|\frac{1}{c} \cdot c\mathbf{x}\| \leq \frac{1}{|c|} \|c\mathbf{x}\| \quad \Rightarrow \quad |c| \, \|\mathbf{x}\| \leq \|c\mathbf{x}\|.$$

   But $0 = |0| \, \|\mathbf{x}\| \leq \|0\mathbf{x}\| = 0$, so this second inequality $|a| \, \|\mathbf{x}\| \leq \|a\mathbf{x}\|$ actually holds for all scalars a. Combined with the first inequality, it implies that $\|a\mathbf{x}\| = |a| \, \|\mathbf{x}\|$ for all scalars a. □

5. **Solution:** If $\mathbf{x} \in \mathbf{Y}$ and $\mathbf{x} \in \mathbf{Y}^\perp$, then $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = 0$, so $\mathbf{x} = \mathbf{0}$ because the derived norm is nondegenerate. Note that $\mathbf{Y} \cap \mathbf{Y}^\perp = \emptyset$ if $\mathbf{0} \notin \mathbf{Y}$. For the second part, apply definition 2.30: for any $\mathbf{y} \in \mathbf{Y}$, we have $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{x} \in \mathbf{Y}^\perp$, so $\mathbf{Y} \subset \left(\mathbf{Y}^\perp\right)^\perp$. □