Multiplication of Short Wavelet Series Using Connection Coefficients*

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Abstract

Given two functions approximable with short wavelet series, we wish to find the short wavelet series representing their product. This can be done by pre-calculating the *connection coefficients* which express the product of two wavelets or scaling functions as a wavelet series. We follow a method suggested by Daubechies and also used by Dahmen, *et al.*, to rapidly compute these coefficients as elements of a matrix which solves a fixed-point problem, and derive some of the formulas and identities satisfied by the coefficients. We estimate the complexity of the connection coefficient multiplication algorithm by counting the number of terms, and then illustrate through a series of graphs how few of these terms are non-negligible.

1 Introduction

We are motivated in our present work by recent speedups in numerical simulations obtained by representing solutions to complicated problems as superpositions of relatively few basic functions. These good basic functions are called *wavelet packets*; they have three useful properties:

- They are almost as well localized in both position and wavenumber as the Heisenberg uncertainty inequality allows;
- They can be assembled into orthogonal bases;
- They come equipped with fast well-conditioned transformations: to compute N expansion coefficients of a function costs only $O(N \log N)$ operations.

The wavelet packet approximation scheme is nonlinear, since the function choice depends upon the solution at each time instant. It works by keeping only those component functions with significant amplitudes; the others are discarded. Making this choice to minimize a description length or information cost criterion is called a *best basis* algorithm [3].

Computed simulations of fully-developed turbulence in the two-dimensional Navier–Stokes equation (2D-NSE) provide an example [7, 19]. 2D-NSE simulations on 10^4 to 10^6 grid points indicate that the number of components needed to obtain deterministic predictability for short times is about one-tenth the number of grid points. The number needed for statistical predictability, such as estimates of the slope of a line fitted to the vorticity power spectrum, is about one-hundredth the number of grid points. Turbulence simulations are thus an example "compressible" high-dimensional problem; the relevant features contain far fewer degrees of freedom than the original mathematical model. Transformation into wavelet packet coordinates holds the promise of reducing the number of parameters in large but compressible problems by many orders of magnitude.

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To take advantage of this reduction, it is necessary to perform all of the numerical computations in wavelet packet coordinates. In a series of papers [1, 2, 8, 12, 15, 17, 18], we and others have developed some of the needed tools. The operations which are well understood and efficiently implemented include matrix-vector and matrix-matrix multiplication, numerical differentiation, and certain integral operators. Conspicuously absent from this list is a method for multiplying two functions when each is a superposition of just a few wavelet packets, without re-computing their values at all grid points.

In this article, we consider only wavelet basis functions rather than the more general wavelet packets. Also, we focus our attention on the compactly-supported wavelets of Daubechies and Mallat [5, 13], which are *refinable functions* in the sense that they can be expressed as short linear combinations of dilated and translated versions of themselves. Refinable functions have a cross-scale self-similarity that can be used to compute integrals of their products, and thus to find connection coefficients.

The algebraic properties of refinable functions are well known, and have been heavily exploited in recent papers on wavelets and numerical analysis. Dahmen and Micchelli [4] considered the problem of evaluating integrals of products of refinable functions and their derivatives. Kunoth [10] later implemented the algorithms described in that paper. Latto, Resnikoff, and Tenenbaum [11] also derived a linear system of equations for connection coefficients involving two-scale equations for refinable functions.

Our work supplements theirs and was inspired by a discussion with Ingrid Daubechies. Our goal is to present the algebraic relations satisfied by integrals of refinable functions, to provide a simpler proof of the basic fixed-point theorem behind the iterative numerical algorithm, to implement the algorithm, and then to compute and plot the connection coefficients. The plots will show rapid decay as the scale or position differences grow large, indicating that relatively few connection coefficients contribute significantly to the product. This suggests multiplying wavelet series will have much lower complexity than that which a coarse estimate from the wavelet's support lengths would indicate.

It should be noted that some recent work by Beylkin on wavelet multiplication provides an alternative method of multiplying short series, relying on the paraproduct formula for wavelet expansions [14] and the observation that certain orthonormal wavelets are good approximations to interpolating functions.

2 Definition of Connection Coefficients

2.1 Abstract Orthonormal Bases

Suppose that $\{e_k : k \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mathbf{R})$ consisting of bounded functions. Then the triple product $e_j e_k e_l$ is defined and integrable over $\mathbf{R}$, so we can define the abstract *connection coefficients* of this basis to be the following integrals:

$$\Gamma_{jkl} = \langle e_j, e_k e_l \rangle \stackrel{\text{def}}{=} \int_{\mathbf{R}} \bar{e}_j(t)\, e_k(t) e_l(t)\, dt. \tag{1}$$

These coefficients are used to find the expansion of a product. If $u(t) = \sum_k u_k e_k(t)$ and $v(t) = \sum_l v_l e_l(t)$, then

$$u(t)v(t) = \sum_j \left(\sum_{k,l} \Gamma_{jkl} u_k v_l \right) e_j(t). \tag{2}$$

2.2 Examples

The space $L^2(\mathbf{R})$ is separable and thus has the countable bases needed for the formula in Equations 1 and 2. However, the most useful of these bases require somewhat complicated indexing which can obscure the main ideas. Thus, we will first explore some simpler spaces and their simpler connection coefficients.

2.2.1 Kronecker Basis

The space of sequences ℓ^2 has the *Kronecker basis* $e_k(n) = \delta(n-k)$, where $\delta(x)$ is the Kronecker symbol which is 1 if $x = 0$ and 0 otherwise. The inner product in Equation 1 is a sum rather than an integral, and we see that $\Gamma_{jkl} = \delta(j-k)\delta(j-l)$. The inner summation of Equation 2 simplifies into the pointwise multiplication formula $\sum_{k,l} \Gamma_{jkl} u_k v_l = u_j v_j$.

2.2.2 Fourier Basis

The Fourier basis for $L^2([0,1])$ consists of the functions $e_k(t) = e^{2\pi kt}$, $k \in \mathbf{Z}$. Since this collection of functions is both orthonormal and closed under multiplication, we can easily compute that $\Gamma_{jkl} = \delta(k+l-j)$. We can change variables in the inner summation of Equation 2 to obtain the usual convolution formula $\sum_{k,l} \Gamma_{jkl} u_k v_l = \sum_k u_k v_{j-k}$.

2.2.3 Haar Basis

It has been known since 1910 [9] that there exists a compactly-supported function $\psi = \psi(x)$ having the property that its translates by integers and dilates by powers of two form an orthonormal basis. Namely, the linear span of the following set of orthogonal unit vectors is dense in $L^2(\mathbf{R})$:

$$\{\psi_{sn}(x) \stackrel{\text{def}}{=} 2^{-s/2}\psi(2^{-s}x - n) : s, n \in \mathbf{Z}\}. \tag{3}$$

Haar's "mother" function ψ is simply $\psi(x) = \mathbf{1}(2x) - \mathbf{1}(2x-1)$, where $\mathbf{1}(x)$ is the characteristic or indicator function of the interval $[0,1)$. It generates a basis of functions that are indexed by a pair of integers, so their connection coefficients require six integer indices:

$$\begin{aligned} \Gamma^{str}_{nmk} &\stackrel{\text{def}}{=} \int_{\mathbf{R}} \psi_{sn}(x)\psi_{tm}(x)\psi_{rk}(x)\, dx \qquad (4) \\ &= 2^{-\frac{s+t+r}{2}} \int_{\mathbf{R}} \psi(2^{-s}x - n)\psi(2^{-t}x - m)\psi(2^{-r}x - k)\, dx \end{aligned}$$

For Haar's function, the integral may be evaluated explicitly. The function ψ_{sn} is supported in the interval $2^s[n, n+1)$, and is constant on the left subinterval $2^s[n, n+\frac{1}{2})$, where it is $+2^{-s/2}$, and on the right subinterval $2^s[n+\frac{1}{2}, n+1)$, where it is $-2^{-s/2}$. The product of two Haar functions ψ_{sn} and ψ_{tm} is either zero (if their support intervals are disjoint), or $2^{-s}\mathbf{1}(2^{-s}x - n)$ (if they are equal, *i.e.*, $s = t$ and $n = m$), or $\pm 2^{-\frac{s+t}{2}}\psi_{tm}$ (if their support intervals intersect, and $s > t$). Hence the Haar functions are almost closed under multiplication.

Since ψ is real-valued, the three functions in the connection coefficient integral may be reordered so that $s \geq t \geq r$; in that case,

$$\Gamma^{str}_{nmk} = \begin{cases} 2^{-s/2}, & \text{if } s > t = r \text{ and } m = k \in 2^{s-t}[n, n+\frac{1}{2}) = \mathcal{L}(s-t, n); \\ -2^{-s/2}, & \text{if } s > t = r \text{ and } m = k \in 2^{s-t}[n+\frac{1}{2}, n+1) = \mathcal{R}(s-t, n); \\ 0, & \text{otherwise, with } s \geq t \geq r. \end{cases} \tag{5}$$

Here it has been useful to define $\mathcal{L}(a,j) \stackrel{\text{def}}{=} 2^a[j, j+\frac{1}{2})$ and $\mathcal{R}(a,j) \stackrel{\text{def}}{=} 2^a[j+\frac{1}{2}, j+1)$, the left and right subintervals of the interval $2^a[j, j+1)$, respectively. Another way to write this is $\Gamma^{str}_{nmk} = 2^{-t/2}\delta(t-r)\delta(m-k)\psi_{s-t,n}(m)$.

Accounting for the other orderings of s, t, r, the multiplication formula is thus

$$\begin{aligned} \sum_{t,m,r,k} \Gamma^{str}_{nmk} u_{tm} v_{rk} &= 2^{-s/2} \sum_{t=-\infty}^{s-1} \left[\sum_{m \in \mathcal{L}(s-t,n)} u_{tm}v_{tm} - \sum_{m \in \mathcal{R}(s-t,n)} u_{tm}v_{tm} \right] \\ &+ v_{sn} \sum_{t=s+1}^{\infty} 2^{-t/2} \left[\sum_{n \in \mathcal{L}(t-s,m)} u_{tm} - \sum_{n \in \mathcal{R}(t-s,m)} u_{tm} \right] \\ &+ u_{sn} \sum_{r=s+1}^{\infty} 2^{-r/2} \left[\sum_{n \in \mathcal{L}(r-s,k)} v_{rk} - \sum_{n \in \mathcal{R}(r-s,k)} v_{rk} \right]. \end{aligned} \tag{6}$$

Now $n \in \mathcal{L}(a,j) \iff 2^{-a}n - \frac{1}{2} < j \le 2^{-a}n$ and $n \in \mathcal{R}(a,j) \iff 2^{-a}n - 1 < j \le 2^{-a}n - \frac{1}{2}$. For fixed n and $a > 0$ exactly one of these inequalities will have a solution $j \stackrel{\text{def}}{=} \mathcal{M}(a,n)$, and that solution will be unique. Put $\mathcal{S}(a,n) = +1$ if $n \in \mathcal{L}(a, \mathcal{M}(a,n))$ and $\mathcal{S}(a,n) = -1$ if $n \in \mathcal{R}(a, \mathcal{M}(a,n))$; then the second and third lines of Equation 6 simplify as follows:

$$\begin{aligned}\sum_{t,m,r,k} \Gamma^{str}_{nmk} u_{tm} v_{rk} &= 2^{-s/2} \sum_{t=-\infty}^{s-1} \left[\sum_{m \in \mathcal{L}(s-t,n)} u_{tm} v_{tm} - \sum_{m \in \mathcal{R}(s-t,n)} u_{tm} v_{tm} \right] \\ &\quad + \sum_{t=s+1}^{\infty} 2^{-t/2} \mathcal{S}(t-s,n) \left[u_{t,\mathcal{M}(t-s,n)} v_{sn} + u_{sn} v_{t,\mathcal{M}(t-s,n)} \right]. \end{aligned} \tag{7}$$

Alternatively, one could write $\mathcal{S}(a,n) = \psi(2^a n - \mathcal{M}(a,n))$.

2.3 Discrete Orthonormal Wavelet Basis

The famous generalization of Haar's basis by Daubechies [5] showed that for any positive integer d there exists a compactly-supported function $\psi = \psi(x)$ with d continuous derivatives on $\mathbf{R}$, whose integer translates and power-of-two dilates generate an orthonormal basis for $L^2(\mathbf{R})$ in the same way as the Haar function. The construction begins with the so-called *two-scale equations*:

$$\phi(x) = \sqrt{2} \sum_k h_k \phi(2x-k) \stackrel{\text{def}}{=} H\phi(x); \tag{8}$$

$$\psi(x) = \sqrt{2} \sum_k g_k \phi(2x-k) \stackrel{\text{def}}{=} G\phi(x). \tag{9}$$

Here $\{h_k\}$ and $\{g_k\}$ are two finite sequences of *wavelet filter coefficients* satisfying some orthogonality conditions:

$$\sum_k h_k h_{k+2n} = \delta(n) = \sum_k g_k g_{k+2n}, \qquad \sum_k h_k g_{k+2n} = 0, \qquad g_k = (-1)^{1-k} h_{1-k}. \tag{10}$$

In addition, we may assume that the coefficients are *conventionally normalized* ([16],pp.158–160) so as to satisfy $\sum_k h_{2k} = \sum_k h_{2k+1} = \sum_k g_{2k} = -\sum_k g_{2k+1} = 1/\sqrt{2}$.

Wavelet bases are composed of translated and rescaled versions of the same function:

$$\psi_{sn}(x) \stackrel{\text{def}}{=} 2^{-s/2} \psi(2^{-s}x - n). \tag{11}$$

The two-scale equations produce the connection coefficients through iteration and filtering.

3 Wavelet Connection Coefficients

As in the Haar case, the connection coefficients are labeled by six indices: Γ^{str}_{nmk}. They may be computed from another set of coefficients, namely those derived from the scaling functions of the wavelet basis:

$$\begin{aligned} A^{str}_{nmk} &\stackrel{\text{def}}{=} \int_{\mathbf{R}} \phi_{sn}(x)\phi_{tm}(x)\phi_{rk}(x)\, dx \\ &= 2^{-\frac{s+t+r}{2}} \int_{\mathbf{R}} \phi(2^{-s}x-n)\phi(2^{-t}x-m)\phi(2^{-r}x-k)\, dx \end{aligned} \tag{12}$$

As before, we may suppose that $s \ge t \ge r$, and then by changing variables we obtain

$$A^{str}_{nmk} = 2^{-\frac{r}{2}} A^{s-r,t-r,0}_{nmk} = 2^{-\frac{r}{2}} A^{s-r,t-r,0}_{0,m-2^{s-t}n,k-2^{s-r}n}, \qquad \text{for } n,m,k \in \mathbf{Z} \text{ and } s \ge t \ge r \in \mathbf{Z}. \tag{13}$$

Hence in practice, to obtain A^{str}_{nmk} it suffices to compute

$$A^{ij}_{mk} \stackrel{\text{def}}{=} A^{ij0}_{0mk} = 2^{-\frac{i+j}{2}} \int_{\mathbf{R}} \phi(2^{-i}x)\phi(2^{-j}x-m)\phi(x-k)\, dx, \qquad \text{for } m,k \in \mathbf{Z} \text{ and } i \ge j \in \mathbf{Z}. \tag{14}$$

Likewise, the complete set of connection coefficients Γ^{str}_{nmk} may be obtained from the following numbers, which must be computed for all $m, k \in \mathbf{Z}$ and for all $i \geq j \in \mathbf{Z}$:

$$\Gamma^{ij}_{mk} \stackrel{\text{def}}{=} 2^{-\frac{i+j}{2}} \int_{\mathbf{R}} \psi(2^{-i}x)\psi(2^{-j}x - m)\psi(x-k)\,dx; \qquad \Gamma^{str}_{nmk} = 2^{-\frac{r}{2}} \Gamma^{s-r,t-r}_{m-2^{s-t}n,k-2^{s-r}n}. \tag{15}$$

3.1 Obtaining $\Gamma^{ij}_{\dots}$ from $A^{ij}_{\dots}$ by Filtering

$$\begin{aligned}
\Gamma^{ij}_{nmk} &= 2^{-\frac{i+j}{2}} \int_{\mathbf{R}} \psi(2^{-i}x-n)\psi(2^{-j}x-m)\psi(x-k)\,dx \\
&= \frac{1}{2} 2^{-\frac{i+j}{2}} \int_{\mathbf{R}} \psi(2^{-i}\frac{x}{2}-n)\psi(2^{-j}\frac{x}{2}-m)\psi(\frac{x}{2}-k)\,dx \\
&= \sqrt{2} \sum_{n',m',k'} g_{n'}g_{m'}g_{k'} \left(2^{-\frac{i+j}{2}} \int_{\mathbf{R}} \phi(2^{-i}x-2n-n')\phi(2^{-j}x-2m-m')\phi(x-2k-k')\,dx \right) \\
&= \sqrt{2} \sum_{n',m',k'} g_{n'}g_{m'}g_{k'} A^{ij}_{2n+n',2m+m',2k+k'}.
\end{aligned}$$

This converts to $\Gamma^{ij}_{nmk} = \sqrt{2}G_1G_2G_3A^{ij}_{n,m,k}$ once we define the commuting operators

$$\begin{aligned}
G_1B(n,m,k) &\stackrel{\text{def}}{=} \sum_{n'} g_{n'}B(2n+n',m,k) \\
G_2B(n,m,k) &\stackrel{\text{def}}{=} \sum_{m'} g_{n'}B(n,2m+m',k) \\
G_3B(n,m,k) &\stackrel{\text{def}}{=} \sum_{k'} g_{n'}B(n,m,2k+k').
\end{aligned}$$

3.2 Obtaining $A^{ij}_{\dots}$ from $A_{\dots}$ by Filtering

$$\begin{aligned}
A^{ij}_{nmk} &= 2^{-\frac{i+j}{2}} \int_{\mathbf{R}} \phi(2^{-i}x-n)\phi(2^{-j}x-m)\phi(x-k)\,dx \\
&= 2^{-\frac{i+j}{2}} \int_{\mathbf{R}} \phi(2^{-i+1}\frac{x}{2}-n)\phi(2^{-j}x-m)\phi(x-k)\,dx \\
&= \sum_{n'} h_{n'} \left(2^{-\frac{(i-1)+j}{2}} \int_{\mathbf{R}} \phi(2^{-(i-1)}x-2n-n')\phi(2^{-j}x-m)\phi(x-k)\,dx \right) \\
&= \sum_{n'} h_{n'} A^{i-1,j}_{2n+n',m,k} \stackrel{\text{def}}{=} H_1 A^{i-1,j}_{n,m,k}.
\end{aligned}$$

Iterating i times in the first scale index and j times in the second gives $A^{ij}_{nmk} = H_1^i H_2^j A_{n,m,k}$, using the commuting operators

$$\begin{aligned}
H_1B(n,m,k) &\stackrel{\text{def}}{=} \sum_{n'} h_{n'}B(2n+n',m,k) \\
H_2B(n,m,k) &\stackrel{\text{def}}{=} \sum_{m'} h_{n'}B(n,2m+m',k)
\end{aligned}$$

3.3 Obtaining Γ from A by Filtering

Combining the iterations of H and G yields

$$\Gamma^{ij}_{nmk} = \sqrt{2}G_1G_2G_3H_1^iH_2^jA_{n,m,k}. \tag{16}$$

3.4 Obtaining A as a Fixed Point

Now suppose that $\phi = \phi(t)$ is a normalized fixed point of Equation 8, i.e., $\int \phi(x)\,dx = 1$ and $\|\phi\| = 1$. Then $\{\phi(x-k) : k \in \mathbf{Z}\}$ is a (real) orthonormal system in L^2, though it is not complete, and we can define the following:

$$A_{nmk} \stackrel{\text{def}}{=} A^{000}_{nmk} = \int_{\mathbf{R}} \phi(x-n)\phi(x-m)\phi(x-k)\,dx \tag{17}$$

This quantity will be defined for all triplets of integers, and will vanish whenever some pair of $\{n, m, k\}$ are so different that the supports of the corresponding scaling functions are disjoint.

Now $A_{nmk} = A_{n-k,m-k,0}$, so it suffices to compute the simpler quantity:

$$A(n,m) \stackrel{\text{def}}{=} A_{nm0} = \int_{\mathbf{R}} \phi(x-n)\phi(x-m)\phi(x)\,dx \tag{18}$$

But this is a matrix which satisfies the following analogue of Equation 8:

Theorem 3.1 *Suppose that $h_k = 0$ unless $0 \le k < L$. Then $A(n,m) = 0$ unless $-L < n < L$ and $-L < m < L$. Also, A satisfies the homogeneous fixed-point equation*

$$A(n,m) = \sum_{p,q} \alpha(p,q) A(2n-p, 2m-q), \qquad -L < n, m < L,$$

where

$$\alpha(p,q) \stackrel{\text{def}}{=} \sqrt{2} \sum_{k=0}^{L} h_{k-p} h_{k-q} h_k, \qquad -L < p, q < L.$$

Proof: We change variables $x \leftarrow x/2$ in the integral defining $A(n,m)$, use Equation 8, and interchange the finite summation and the integration:

$$\begin{aligned}
A(n,m) &= \frac{1}{2}\int_{\mathbf{R}} \phi(\frac{x}{2}-n)\phi(\frac{x}{2}-m)\phi(\frac{x}{2})\,dx \\
&= \sqrt{2}\int_{\mathbf{R}} \left(\sum_{n'} h_{n'}\phi(x-2n-n')\right)\left(\sum_{m'} h_{m'}\phi(x-2m-m')\right)\left(\sum_{k'} h_{k'}\phi(x-k')\right) dx \\
&= \sqrt{2} \sum_{n',m',k'} h_{n'}h_{m'}h_{k'} \int_{\mathbf{R}} \phi(x-2n-n')\phi(x-2m-m')\phi(x-k')\,dx \\
&= \sqrt{2} \sum_{n',m',k'} h_{n'}h_{m'}h_{k'} \int_{\mathbf{R}} \phi(x+k'-2n-n')\phi(x+k'-2m-m')\phi(x)\,dx \\
&= \sqrt{2} \sum_{n',m',k'} h_{n'}h_{m'}h_{k'} A(2n+n'-k', 2m+m'-k'). \\
&= \sum_{p,q} \left(\sqrt{2}\sum_{k} h_{k-p}h_{k-q}h_k\right) A(2n-p, 2m-q).
\end{aligned}$$

At the last step we have substituted $k' \leftarrow k$, $n' \leftarrow k-p$, and $m' \leftarrow k-q$.

To get the support rectangle, note that if h_k vanishes for $k < 0$ or $k \ge L$, then $\phi(x)$ will be supported in $[0, L]$ (see [6], p.167) and $A(n,m)$ will vanish for $|n| \ge L$ or $|m| \ge L$. □

If we put $\hat{f}(\xi,\eta) \stackrel{\text{def}}{=} \sum_{n,m\in\mathbf{Z}} f(n,m)e^{-2\pi i n\xi}e^{-2\pi i m\eta}$ for any double sequence $f = f(n,m)$, then we can write

$$\begin{aligned}
\hat{\alpha}(\xi,\eta) &= \sum_{n,m\in\mathbf{Z}} \sqrt{2}\sum_{k=0}^{L-1} h_k h_{k-n} h_{k-m} e^{-2\pi i n\xi} e^{-2\pi i m\eta} \\
&= \sqrt{2}\left(\sum_{k=0}^{L-1} h_k e^{-2\pi i k(\xi+\eta)}\right)\left(\sum_{p=0}^{L-1} h_p e^{2\pi i p\xi}\right)\left(\sum_{q=0}^{L-1} h_q e^{2\pi i q\eta}\right) \\
&\stackrel{\text{def}}{=} \sqrt{2}\, m(\xi+\eta)m(-\xi)m(-\eta).
\end{aligned} \tag{19}$$

The second equality follows from the substitutions $n \leftarrow k-p$ and $m \leftarrow k-q$. The sums in parentheses are trigonometric polynomials, and we adopt the convention of Reference [16], p.176, when defining the filter multiplier m. Note that $\hat{\alpha}(0,0)=4$, since $m(0)=\sqrt{2}$. Also, $\hat{\alpha}(\xi,\eta)=\hat{\alpha}(\eta,\xi)$ since $\alpha(n,m)=\alpha(m,n)$. Finally, if the filter coefficients are real-valued, then $m(-\xi)=\bar{m}(\xi)$ and $\hat{\alpha}(\xi,-\xi)=\hat{\alpha}(\xi,0)=\hat{\alpha}(0,\xi)=2|m(\xi)|^2$.

Remark. Replacing $\{h_k\} \leftarrow \{g_k\}$ and $\phi \leftarrow \psi$ in the arguments above gives the following formula:

$$\Gamma^{00}_{nm} = \sum_{p,q} \beta(p,q) A(2n-p, 2m-q), \qquad -L < n,m < L, \tag{20}$$

where

$$\beta(p,q) \stackrel{\text{def}}{=} \sqrt{2} \sum_{k=0}^{L} g_{k-p} g_{k-q} g_k, \qquad -L < p,q < L. \tag{21}$$

The conjugate quadrature filter relationship between $\{h_k\}$ and $\{g_k\}$ implies that

$$\hat{\beta}(\xi,\eta) = -\sqrt{2}\, \bar{m}(\frac{1}{2}+\xi+\eta)\bar{m}(\frac{1}{2}-\xi)\bar{m}(\frac{1}{2}-\eta). \tag{22}$$

3.4.1 Existence and Uniqueness

From the filter coefficients $\{h_k\}$ in the guise of α, we can define an operator T on double sequences:

$$Tf(n,m) \stackrel{\text{def}}{=} \sum_{p,q} \alpha(p,q) f(2n-p, 2m-q) = \sum_{p,q} \alpha(2n-p, 2m-q) f(p,q). \tag{23}$$

The matrix $A = A(n,m)$ is a fixed point of this operator and is completely determined by it except for normalization. Using Lemma 5.13 in Reference [16], p.178, we can rewrite the fixed point equation as follows:

$$\begin{aligned} \hat{A}(\xi,\eta) = \widehat{TA}(\xi,\eta) &= \frac{1}{4}\hat{\alpha}(\frac{\xi}{2},\frac{\eta}{2})\hat{A}(\frac{\xi}{2},\frac{\eta}{2}) + \frac{1}{4}\hat{\alpha}(\frac{\xi}{2}+\frac{1}{2},\frac{\eta}{2}+\frac{1}{2})\hat{A}(\frac{\xi}{2}+\frac{1}{2},\frac{\eta}{2}+\frac{1}{2}) \\ &\quad + \frac{1}{4}\hat{\alpha}(\frac{\xi}{2}+\frac{1}{2},\frac{\eta}{2})\hat{A}(\frac{\xi}{2}+\frac{1}{2},\frac{\eta}{2}) + \frac{1}{4}\hat{\alpha}(\frac{\xi}{2},\frac{\eta}{2}+\frac{1}{2})\hat{A}(\frac{\xi}{2},\frac{\eta}{2}+\frac{1}{2}). \end{aligned}$$

This is rather complicated, and it is easier to compute the action of the adjoint operator:

$$T^*g(p,q) \stackrel{\text{def}}{=} \sum_{n,m} \bar{\alpha}(2n-p, 2m-q) g(n,m); \qquad \widehat{T^*g}(\xi,\eta) = \overline{\hat{\alpha}(\xi,\eta)}\hat{g}(2\xi,2\eta). \tag{24}$$

Theorem 3.2 *The equation $A = TA$ with normalization $\sum_{n,m} A(n,m) = 1$ has a unique solution.*

Proof: We first show that the normalization is preserved by T:

$$\sum_{n,m} Tf(n,m) = \sum_{n,m}\sum_{p,q} \alpha(2n-p,2m-q) f(p,q) = \sum_{p,q} \left(\sqrt{2} \sum_{n,m,k} h_{k-2n+p} h_{k-2m+q} h_k \right) f(p,q).$$

Now $\sum_n h_{2n} = \sum_n h_{2n+1} = 1/\sqrt{2}$, so the sum over n,m,k inside the parentheses equals 1 for all p,q. Hence the right hand side simplifies to $\sum_{p,q} f(p,q)$.

To show the existence of $A(n,m)$, start with any fixed $f = f(n,m)$ satisfying $\sum_{n,m} f(n,m) = 1$, define $A_k = T^k f$ for $k = 1,2,...$, and show that $A_k \to A$ as $k \to \infty$. Then A will solve $A = TA$ and $\sum_{n,m} A(n,m) = 1$. But if $g = g(n,m)$ is any double sequence, then $\langle g, T^k f\rangle = \langle T^{*k} g, f\rangle$ which by Plancherel's theorem is equal to the inner product between the Fourier transforms:

$$\begin{aligned} \langle \widehat{T^{*k}g}, \hat{f}\rangle &= \int_0^1\int_0^1 \left[\prod_{j=0}^{k-1} \hat{\alpha}(2^j\xi, 2^j\eta)\right] \overline{\hat{g}(2^k\xi, 2^k\eta)} \hat{f}(\xi,\eta)\, d\xi d\eta \\ &= \frac{1}{2^{2k}} \int_0^{2^k}\int_0^{2^k} \left[\prod_{j=1}^{k} \hat{\alpha}(\frac{\xi}{2^j}, \frac{\eta}{2^j})\right] \overline{\hat{g}(\xi,\eta)} \hat{f}(\frac{\xi}{2^k}, \frac{\eta}{2^k})\, d\xi d\eta \end{aligned}$$

Now for each fixed (ξ,η), $\frac{1}{2^{2k}}\prod_{j=1}^{k}\hat{\alpha}(\frac{\xi}{2^j},\frac{\eta}{2^j}) \to \hat{\phi}(\xi+\eta)\hat{\phi}(-\eta)\hat{\phi}(-\xi)$ and $\hat{f}(\frac{\xi}{2^k},\frac{\eta}{2^k}) \to 1$ as $k\to\infty$. This convergence is uniform on compact subsets of $\mathbf{R}^2$, and since the filter $\{h_k\}$ determines a regular scaling function ϕ, $\hat{\phi}$ vanishes to sufficiently high order at ∞ for the integrand to converge in $L(\mathbf{R}^2)$ as well. Then, since $\hat{g}=\hat{g}(\xi,\eta)$ is 1-periodic in both ξ and η, the integration breaks up over the $2^k\times 2^k$ periods:

$$
\begin{aligned}
\langle \widehat{T^{*k}g}, \hat{f}\rangle &\approx \sum_{p=0}^{2^k-1}\sum_{q=0}^{2^k-1}\int_0^1\int_0^1 \hat{\phi}(\xi+p+\eta+q)\hat{\phi}(-\eta-p)\hat{\phi}(-\xi-q)\overline{\hat{g}(\xi,\eta)}\,d\xi d\eta \\
&\to \int_0^1\int_0^1 \hat{\phi}_1(\xi+\eta)\hat{\phi}_1(-\eta)\hat{\phi}_1(-\xi)\overline{\hat{g}(\xi,\eta)}\,d\xi d\eta, \qquad \text{as } k\to\infty,
\end{aligned}
$$

where $\hat{\phi}_1(\xi) \stackrel{\text{def}}{=} \sum_{j\in\mathbf{Z}}\hat{\phi}(\xi+j)$ is the 1-periodization of $\hat{\phi}$. Thus $A_k\to A$ in $\ell^2(\mathbf{Z}^2)$ and the limit sequence satisfies $\hat{A}(\xi,\eta)=\hat{\phi}_1(\xi+\eta)\hat{\phi}_1(-\eta)\hat{\phi}_1(-\xi)$.

Finally, to show uniqueness, suppose that $A'=TA'$ with $\sum_{n,m}A'(n,m)=1$. Starting with $f=A'$ in the iteration above gives $A'=T^nA'\to A$ as $n\to\infty$. □

3.4.2 Example: Haar Wavelet Basis

For this wavelet basis, we have $h_0=h_1=\frac{1}{\sqrt{2}}$ and $h_k=0$ if $k\notin\{0,1\}$. Thus $L=2$, and

$$
\alpha = \frac{1}{2}\left\{\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{array}\right\}; \qquad A(n,m)=\delta(n)\delta(m) \tag{25}
$$

3.4.3 Example: Coifman 12 Wavelet Basis

For this wavelet basis, we have $L=12$ and $\{h_k : 0\le k<12\}$ is the following table of values:

$$
\begin{array}{lll}
\{1.6387336463179785\times10^{-2}, & -4.1464936781966485\times10^{-2}, & -6.7372554722299874\times10^{-2}, \\
3.8611006682309290\times10^{-1}, & 8.1272363544960613\times10^{-1}, & 4.1700518442377760\times10^{-1}, \\
-7.6488599078264594\times10^{-2}, & -5.9434418646471240\times10^{-2}, & 2.3680171946876750\times10^{-2}, \\
5.6114348193659885\times10^{-3}, & -1.8232088709100992\times10^{-3}, & -7.2054944536811512\times10^{-4}\}
\end{array}
$$

These filter coefficients define the following α, whose entries are multiplied by 1000 and truncated to integers for display purposes:

0	0	−1	−1	0	0	−1	−1	−1	0	0	−1	0	0	0	0	0	0	0	0	0	0	0
0	0	−1	−1	0	0	−1	−1	0	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0
−1	−1	0	0	−1	−1	0	0	0	0	0	−1	−1	0	0	0	0	0	0	0	0	0	0
−1	−1	0	0	−1	−1	0	0	0	−1	−1	−1	0	0	−1	0	0	0	0	0	0	0	0
0	0	−1	−1	0	0	−1	−2	−2	−2	−2	−1	−1	0	0	−1	0	0	0	0	0	0	0
0	0	−1	−1	0	−1	−1	−1	1	1	0	−1	−2	−1	0	−1	−1	0	0	0	0	0	0
−1	−1	0	0	−1	−1	3	8	8	10	14	9	0	−2	0	0	−1	0	0	0	0	0	0
−1	−1	0	0	−2	−1	8	7	−9	−21	−12	11	14	1	−2	0	0	0	0	0	0	0	0
−1	0	0	0	−2	1	8	−9	−45	−65	−76	−61	−12	10	1	−2	0	−1	0	−1	0	0	0
0	0	0	−1	−2	1	10	−21	−65	32	122	8	−76	−21	8	−1	−1	−1	−1	−1	−1	0	0
0	−1	0	−1	−2	0	14	−12	−76	122	549	543	122	−65	−9	8	−1	0	−1	0	−1	0	0
−1	−1	−1	−1	−1	−1	9	11	−61	8	543	941	549	32	−45	7	3	−1	0	0	0	0	0
0	0	−1	0	−1	−2	0	14	−12	−76	122	549	543	122	−65	−9	8	−1	0	−1	0	−1	0
0	0	0	0	0	−1	−2	1	10	−21	−65	32	122	8	−76	−21	8	−1	−1	−1	−1	−1	−1
0	0	0	−1	0	0	0	−2	1	8	−9	−45	−65	−76	−61	−12	10	1	−2	0	−1	0	−1
0	0	0	0	−1	−1	0	0	−2	−1	8	7	−9	−21	−12	11	14	1	−2	0	0	0	0
0	0	0	0	0	−1	−1	0	0	−1	−1	3	8	8	10	14	9	0	−2	0	0	−1	0
0	0	0	0	0	0	0	0	−1	−1	0	−1	−1	−1	1	1	0	−1	−2	−1	0	−1	−1
0	0	0	0	0	0	0	0	0	−1	−1	0	0	−1	−2	−2	−2	−2	−1	−1	0	0	−1
0	0	0	0	0	0	0	0	−1	−1	0	0	−1	−1	0	0	0	−1	−1	−1	0	0	−1
0	0	0	0	0	0	0	0	0	−1	−1	0	0	−1	−1	0	0	0	0	0	−1	−1	0
0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	0	0	−1	−1	0	0	−1	−1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	0	0	−1	−1	−1	0	0	−1

Here the origin $m=k=0$ of $\alpha(m,k)$ is at the center, m increases downwards, and k increases to the right as in the convention for matrices. The corresponding fixed point A is plotted below in Figure 3.

4 Numerical Examples

4.1 Procedure

We start with a filter sequence $\{h_k\}$ taken from the list of filters in Reference [16], which are known to have smooth scaling function limits. We restrict our attention to the Daubechies filters of lengths 4, 6, 10, and 12, and the Coifman filters of lengths 6 and 12. We compute the kernel α of the fixed-point problem and then iterate from the elementary double sequence $A(m,k) = 1 \iff m = k = 0$ until the maximum change per iteration in a coefficient of A falls below 10^{-6}.

To get the other scaling and connection coefficients, we apply the filter operators G_1, G_2, G_3, H_1, and H_2 as needed.

4.2 Space and Time Requirements

Once the low-pass filter $\{h_n : n = 0, 1, ..., L-1\}$ is chosen, it is known that $A(m,k)$ will vanish for $|m| \geq L$ or $|k| \geq L$, so it is only necessary to allocate $(2L-1) \times (2L-1)$ memory locations to hold A. Likewise, $\alpha(m,k)$ fits into a $(2L+1) \times (2L+1)$ array.

Each application of an operator G_1, G_2, G_3, H_1, or H_2 costs L operations per output coefficient, since we sum over the L non-zero coefficients of either $\{h_n\}$ or $\{g_n\}$. The number of coefficients we need to compute grows, however. For filters supported on $\{0, 1, \ldots, L-1\}$ and fixed $i, j \geq 0$, the matrices $A^{ij}(m,k)$ and $\Gamma^{ij}(m,k)$ will vanish outside $-L < m < 2^{i-j}L$ and $-L < k < 2^{i}L$. However, as the plots below will show, many of the coefficients with indices in this range are negligible.

4.3 Graphs

In the following figures, we plot level lines of the logarithm of the absolute value of scalling and connection coefficients. The origin $m = k = 0$ is always at the center of the square. The graphs are oriented such that m increases to the right and k increases upwards as in the convention for xy plots in the first quadrant.

4.3.1 Graphs of $A(m,k) = \int \varphi(x)\varphi(x-m)\varphi(x-k)\, dx$ for filters D4, D6, D10, D12, C6, and C12

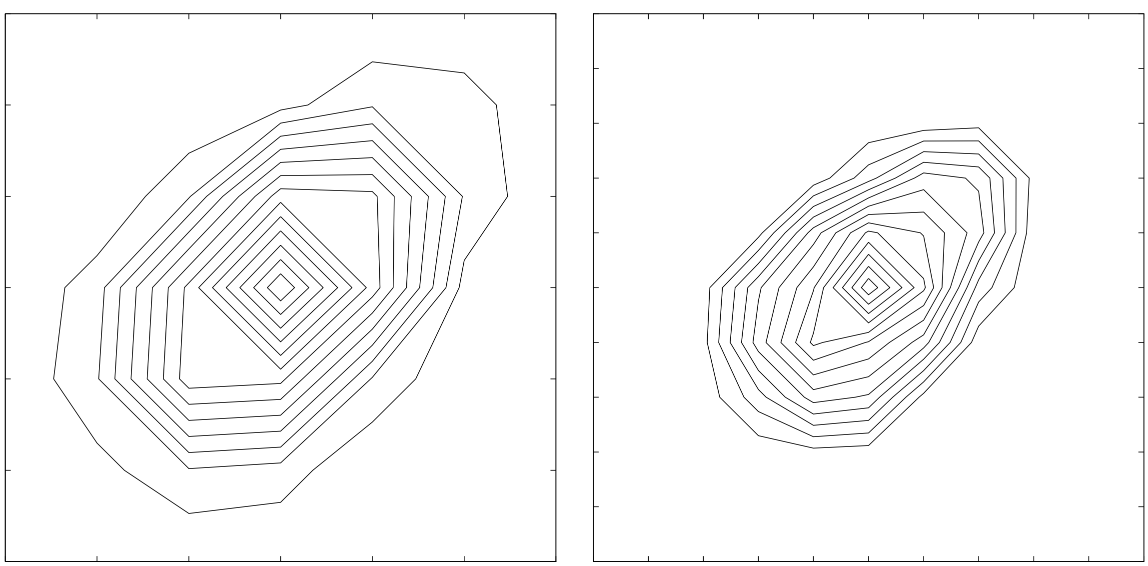

Figure 1: $A(m,k)$ for Daubechies scaling functions of support 4 and 6

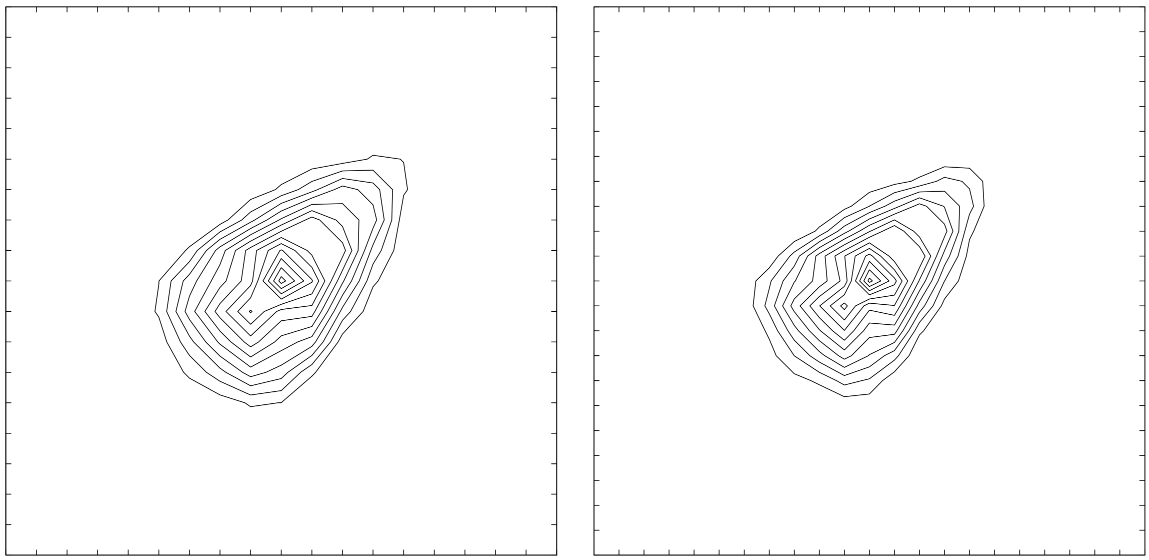

Figure 2: $A(m,k)$ for Daubechies scaling functions of support 10 and 12

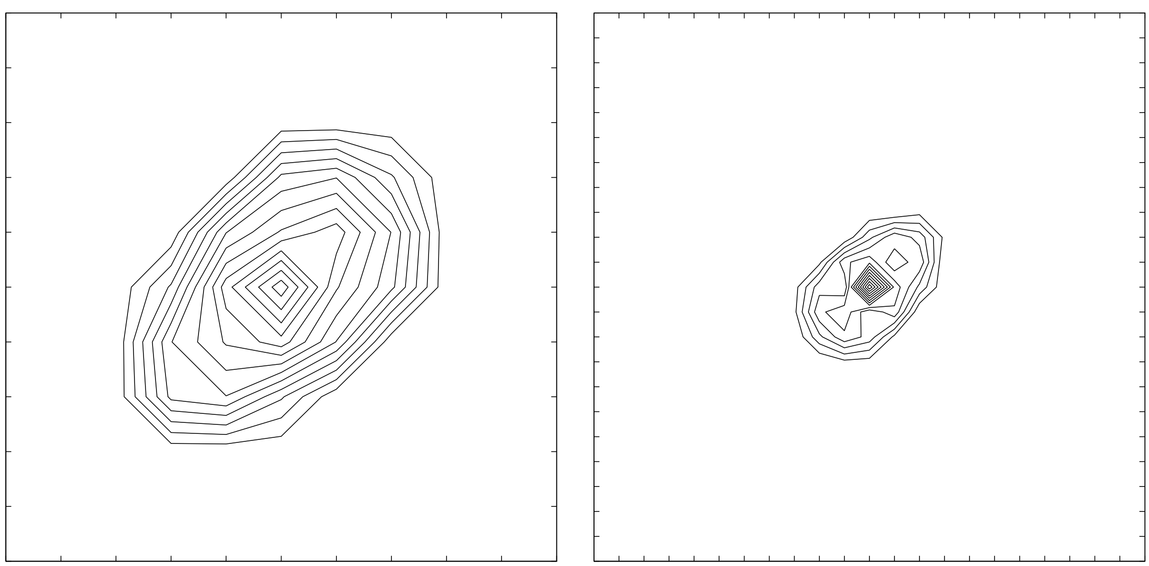

Figure 3: $A(m,k)$ for Coifman scaling functions of support 6 and 12

4.3.2 Graphs of $\Gamma(m,k) = \int \psi(x)\psi(x-m)\psi(x-k)\ dx$ for filters D4, D6, D10, D12, C6, and C12

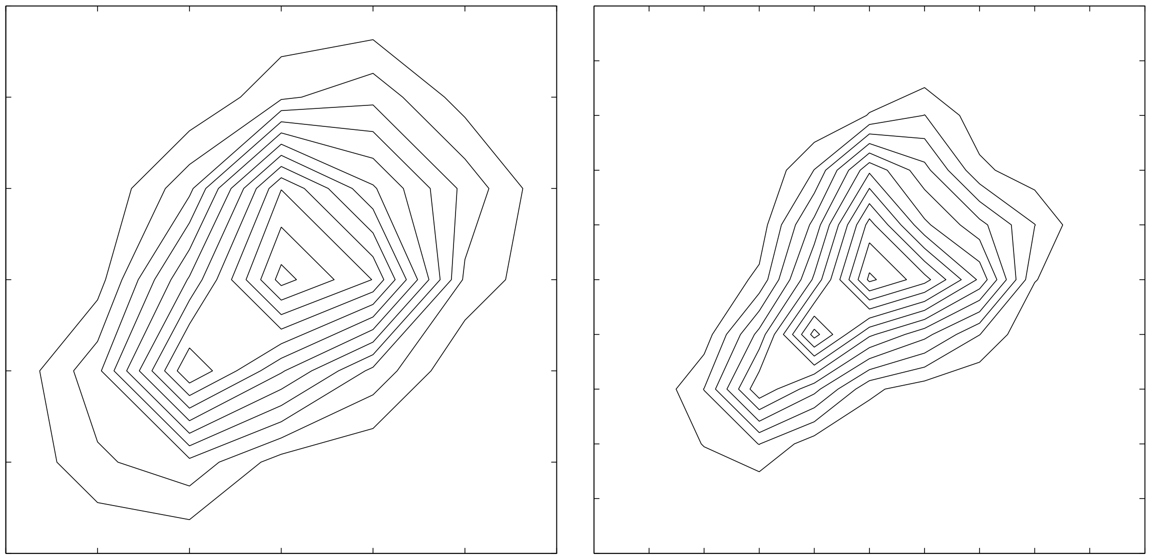

Figure 4: $\Gamma(m,k)$ for Daubechies wavelets of support 4 and 6

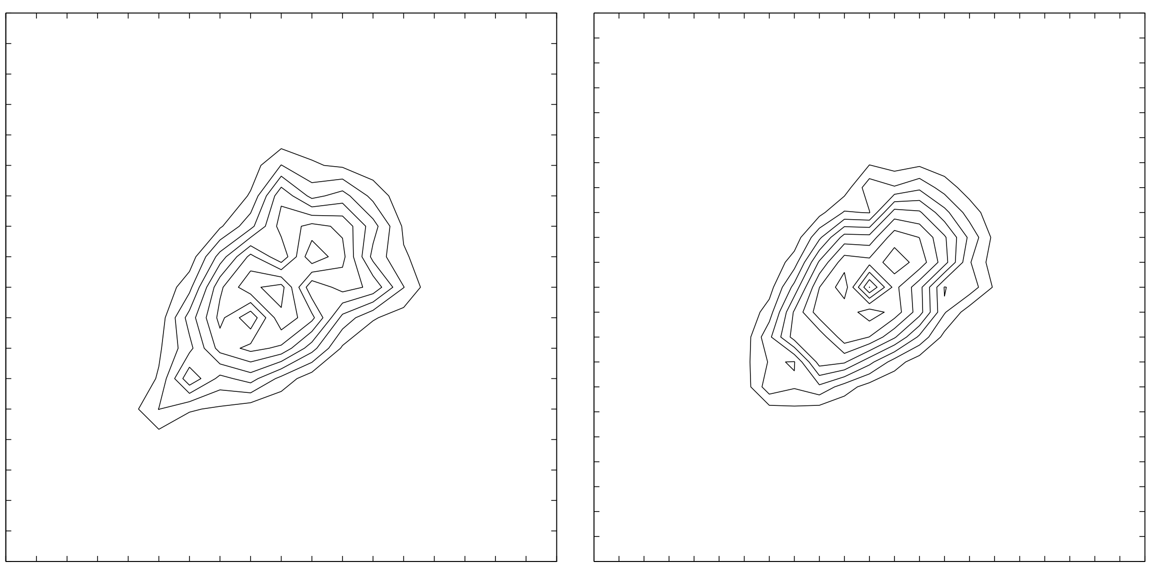

Figure 5: $\Gamma(m,k)$ for Daubechies wavelets of support 10 and 12

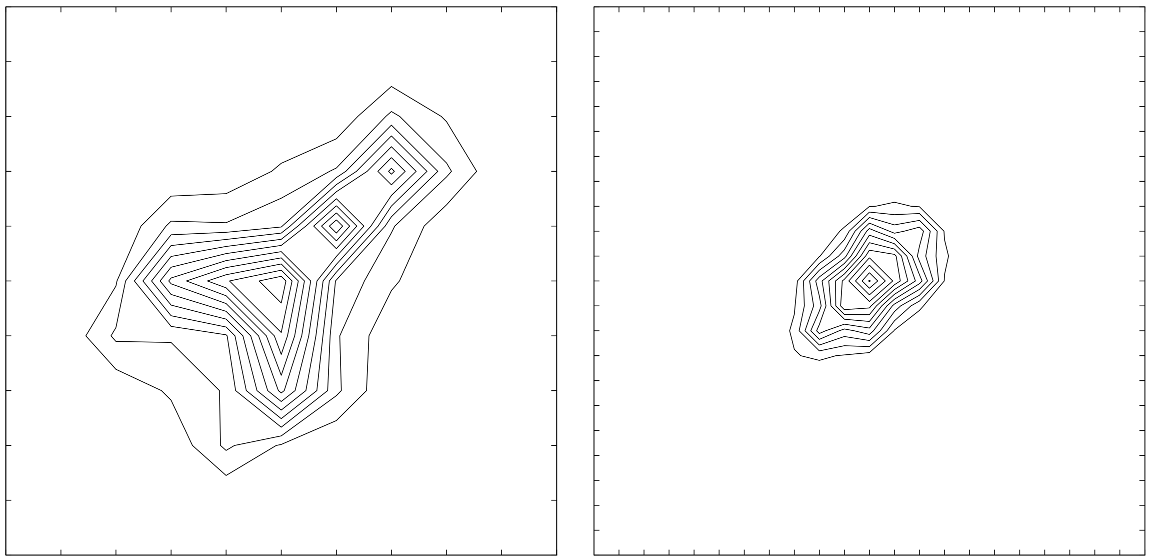

Figure 6: $\Gamma(m,k)$ for Coifman wavelets of support 6 and 12

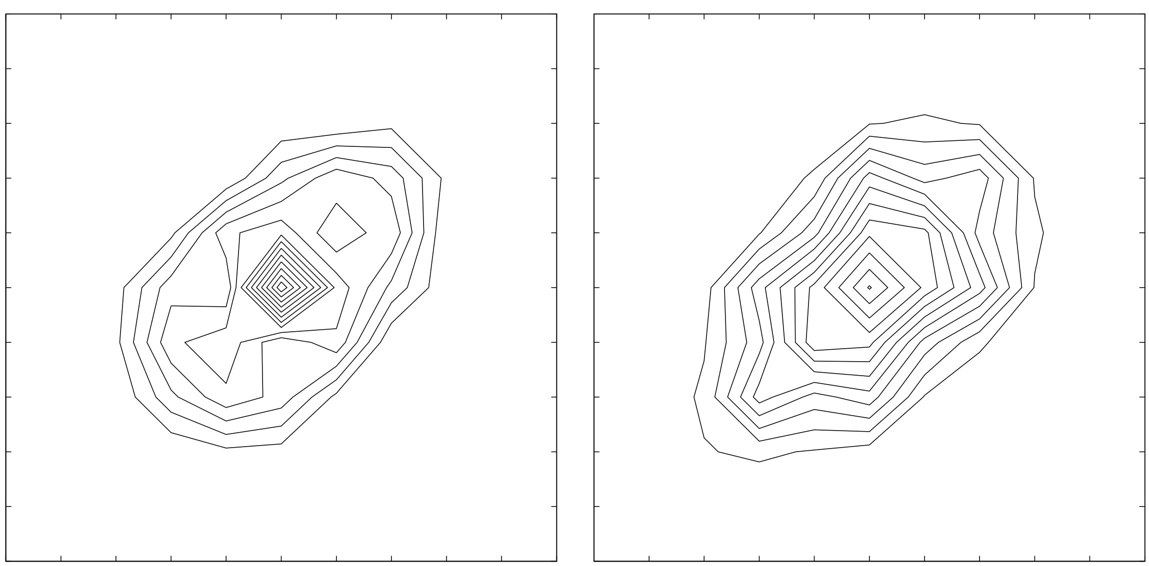

Figure 7: Close-up of $\Gamma(m,k)$ for Coifman scaling function and wavelet of order 12

4.3.3 Graphs of $A^{i,j}(m,k) = 2^{-\frac{i+j}{2}} \int \varphi(\frac{x}{2^i})\varphi(\frac{x}{2^j} - m)\varphi(x-k)\ dx$ for filters D6, C6, and C12

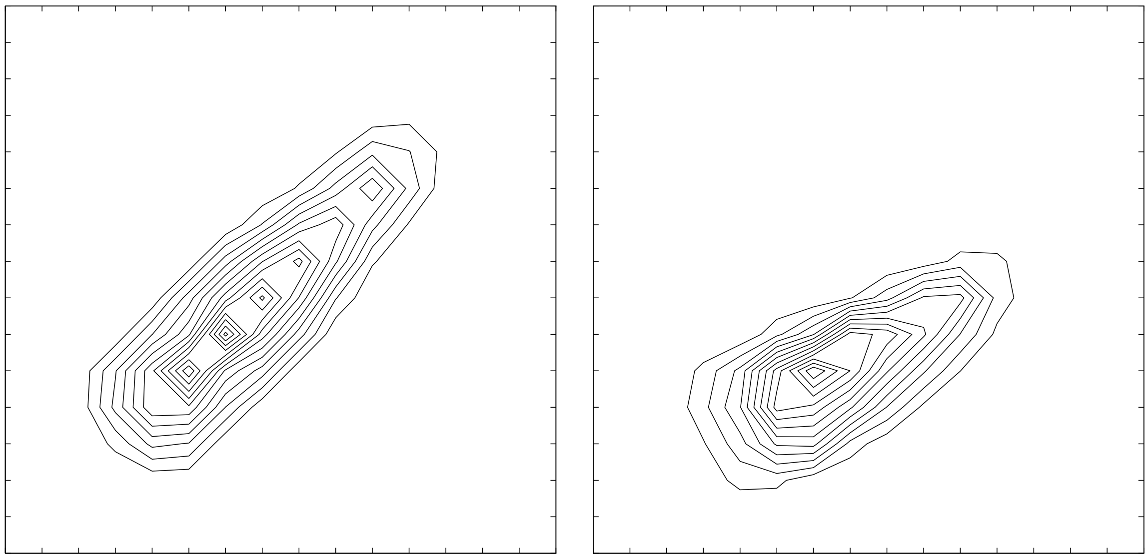

Figure 8: $A^{1,0}(m,k)$ and $A^{1,1}(m,k)$ for Daubechies 6

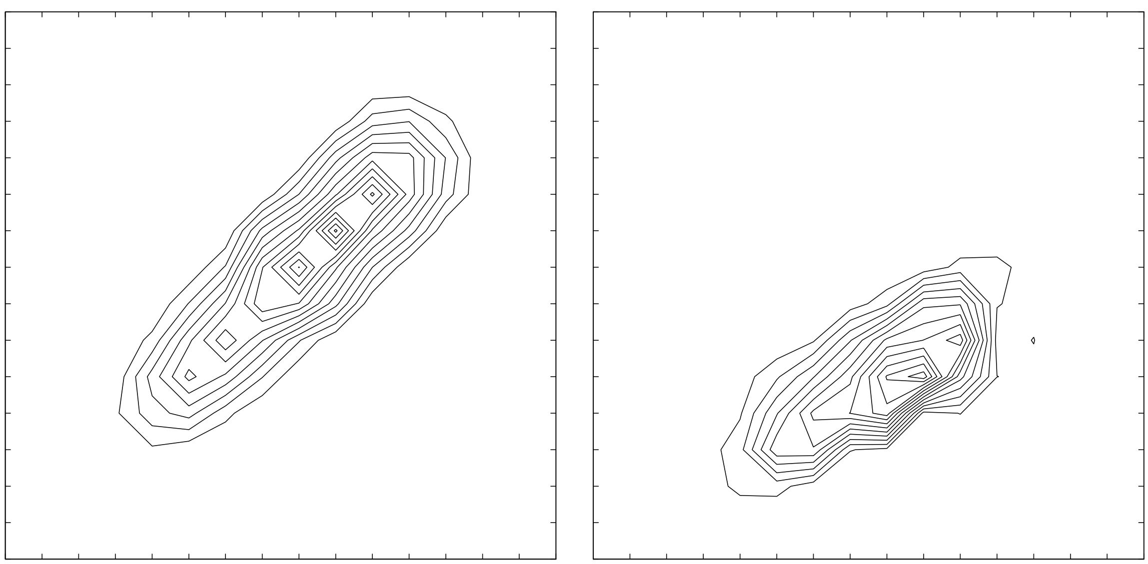

Figure 9: $A^{1,0}(m,k)$ and $A^{1,1}(m,k)$ for Coifman 6

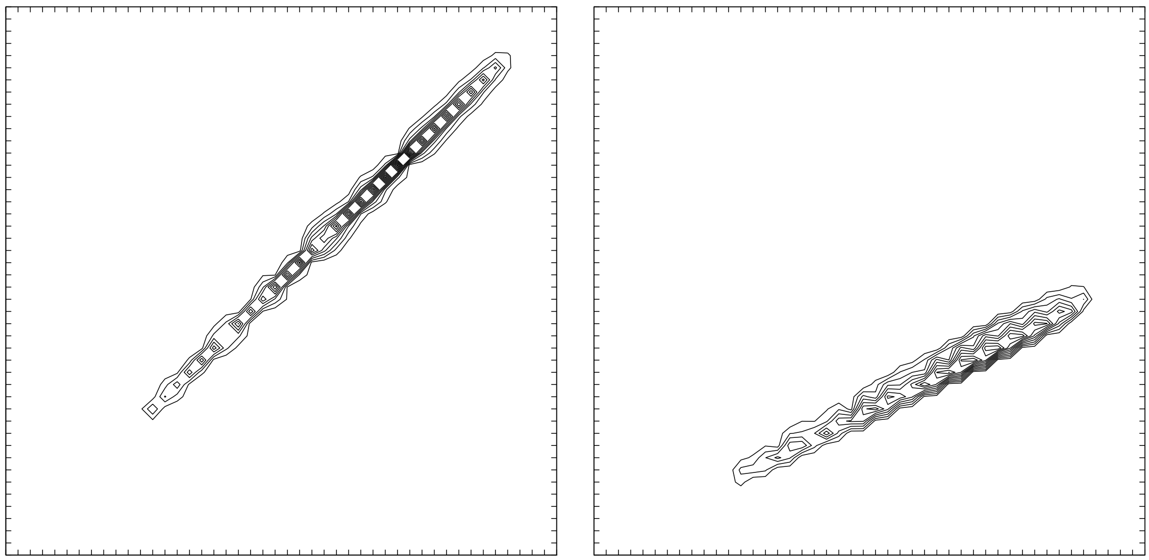

Figure 10: $A^{3,0}(m,k)$ and $A^{3,1}(m,k)$ for Coifman 6

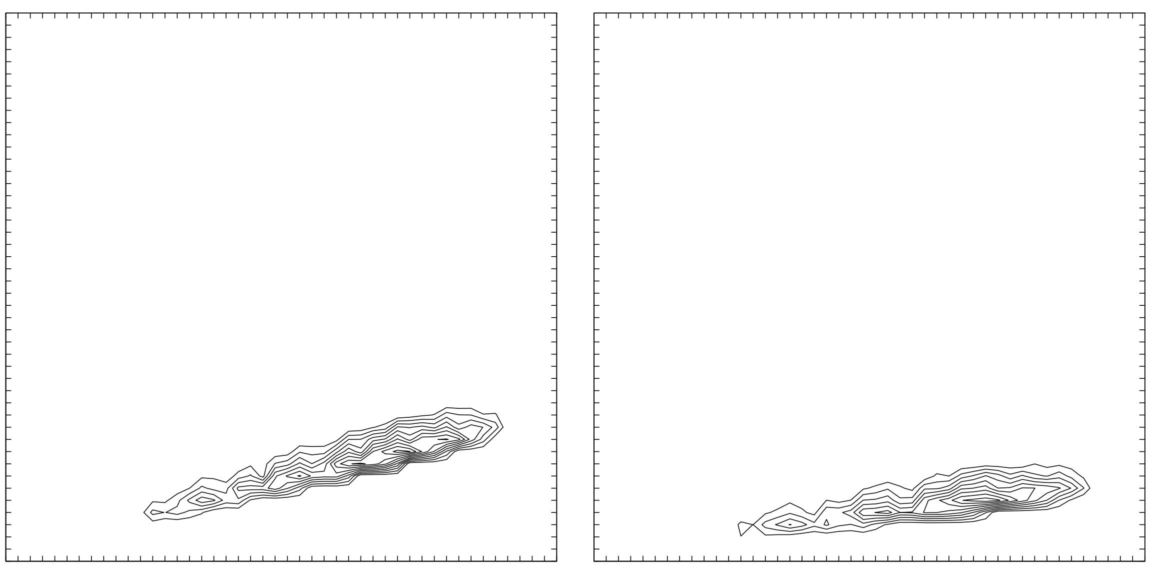

Figure 11: $A^{3,2}(m,k)$ and $A^{3,3}(m,k)$ for Coifman 6

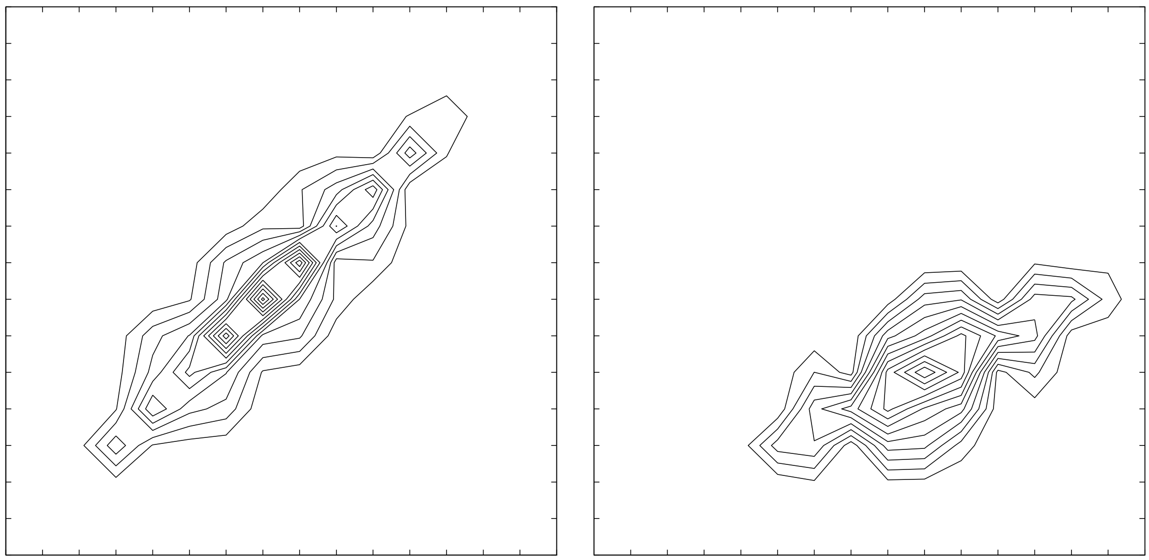

Figure 12: Close-ups of $A^{1,0}(m,k)$ and $A^{1,1}(m,k)$ for Coifman 12

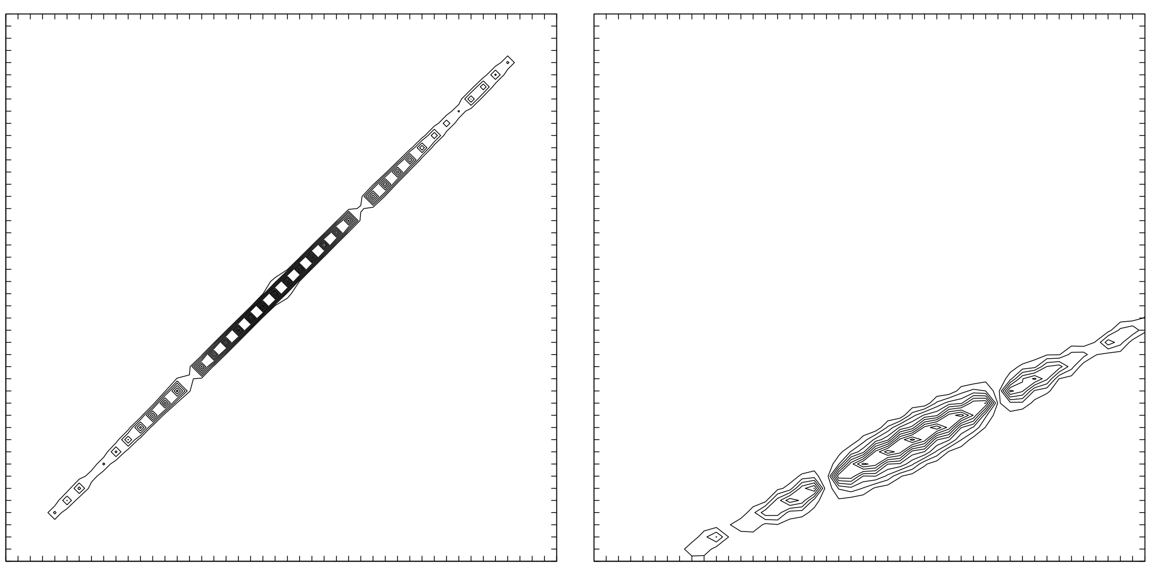

Figure 13: Close-ups of $A^{3,0}(m,k)$ and $A^{3,1}(m,k)$ for Coifman 12

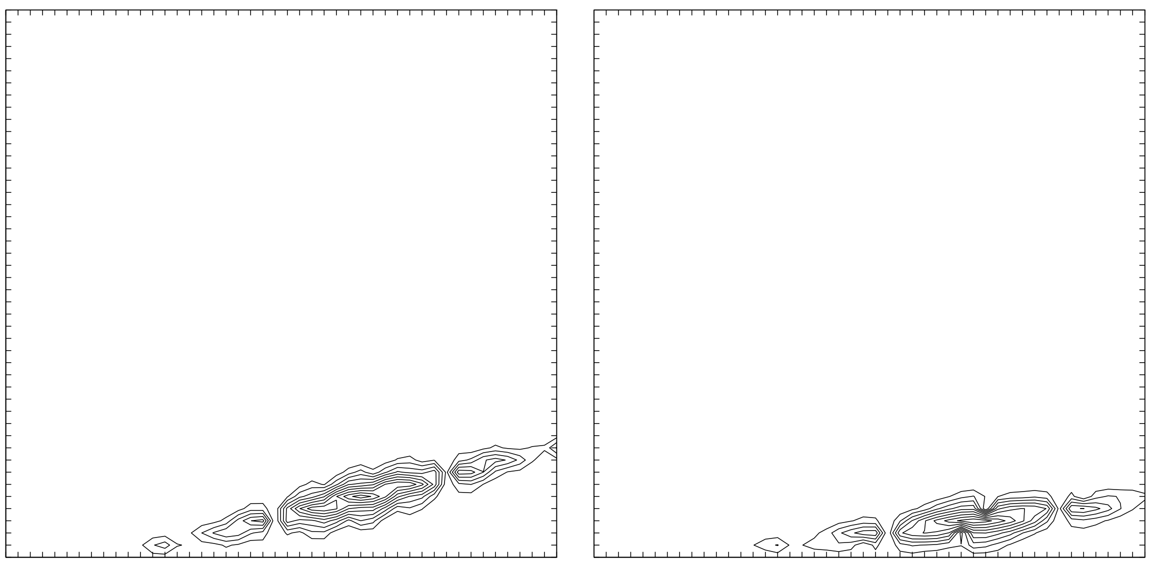

Figure 14: Close-ups of $A^{3,2}(m,k)$ and $A^{3,3}(m,k)$ for Coifman 12

4.3.4 Graphs of $\Gamma^{i,j}(m,k) = 2^{-\frac{i+j}{2}} \int \psi(\frac{x}{2^i})\psi(\frac{x}{2^j} - m)\psi(x-k)\ dx$ for filters D6, C6, and C12.

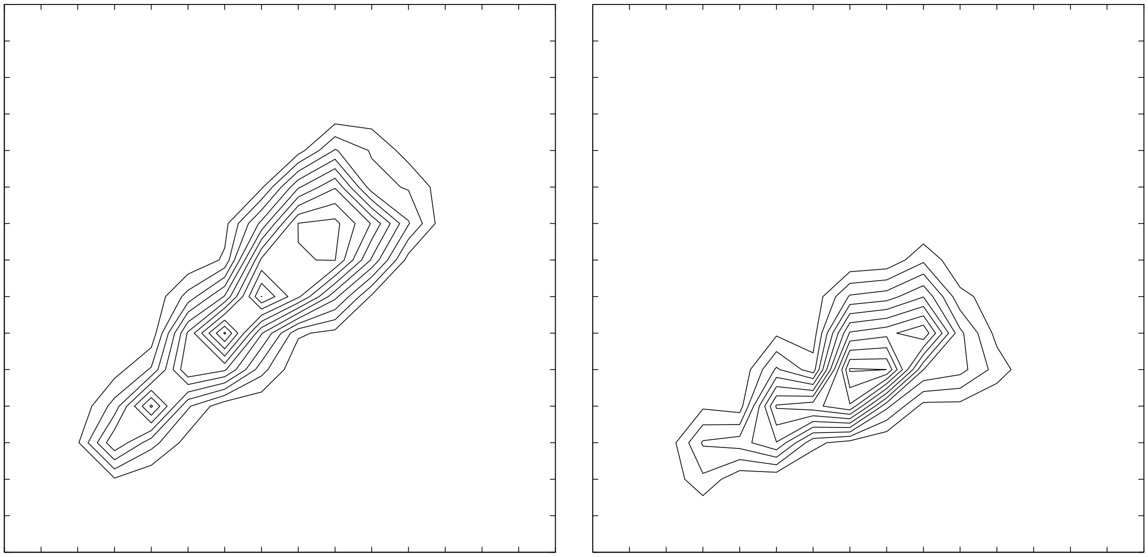

Figure 15: $\Gamma^{1,0}(m,k)$ and $\Gamma^{1,1}(m,k)$ for Daubechies 6

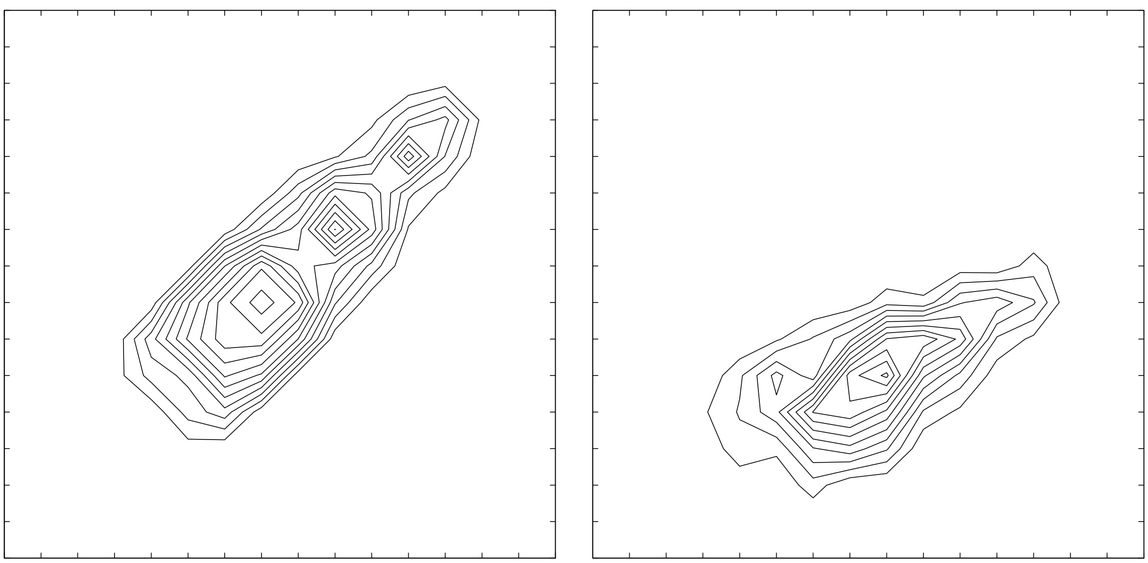

Figure 16: $\Gamma^{1,0}(m,k)$ and $\Gamma^{1,1}(m,k)$ for Coifman 6

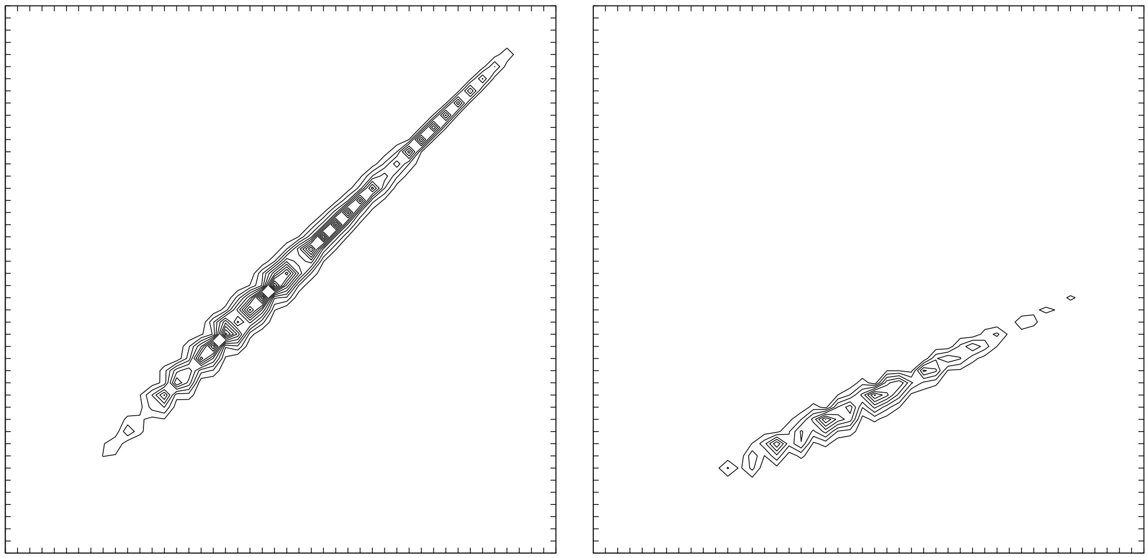

Figure 17: $\Gamma^{3,0}(m,k)$ and $\Gamma^{3,1}(m,k)$ for Coifman 6

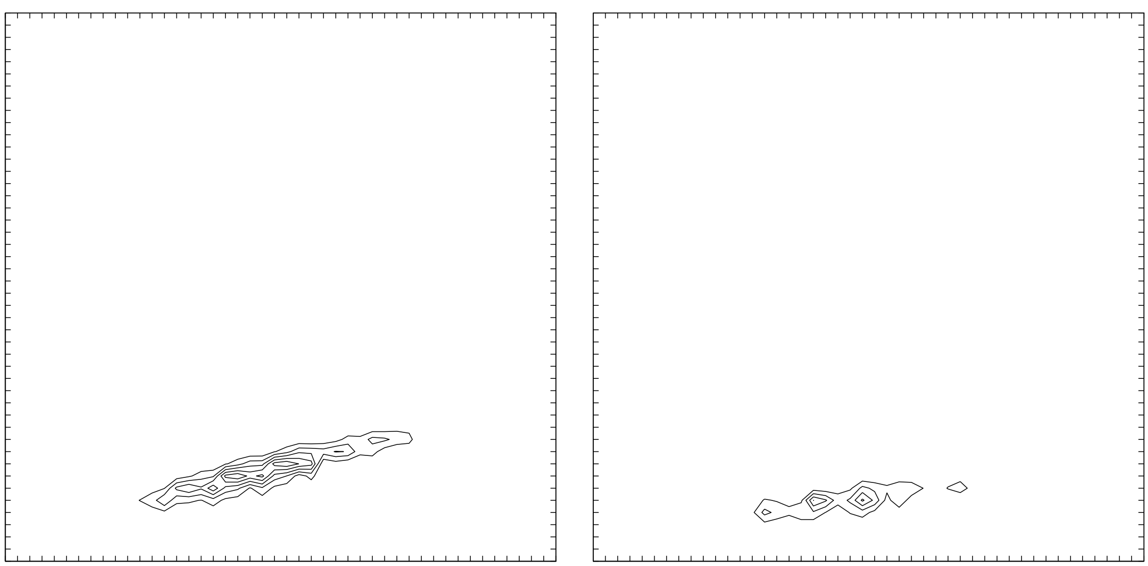

Figure 18: $\Gamma^{3,2}(m,k)$ and $\Gamma^{3,3}(m,k)$ for Coifman 6

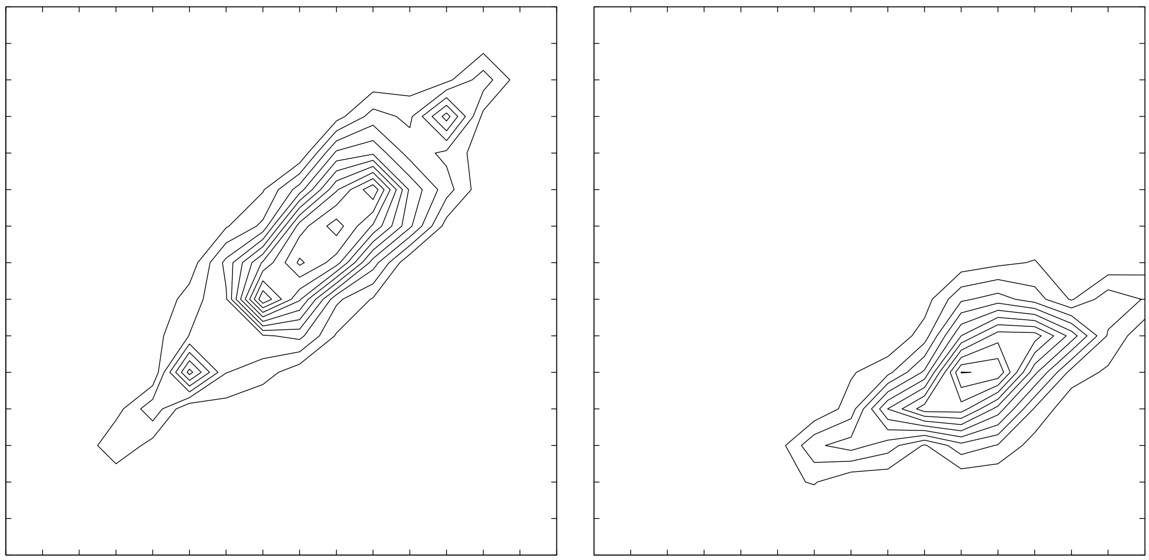

Figure 19: Close-ups of $\Gamma^{1,0}(m,k)$ and $\Gamma^{1,1}(m,k)$ for Coifman 12

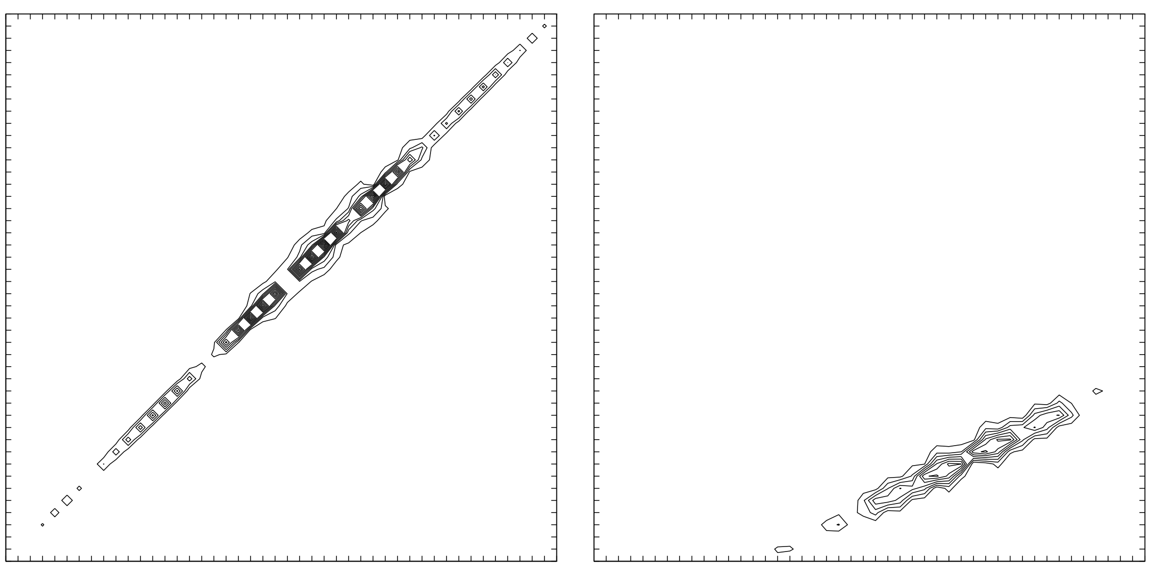

Figure 20: Close-ups of $\Gamma^{3,0}(m,k)$ and $\Gamma^{3,1}(m,k)$ for Coifman 12

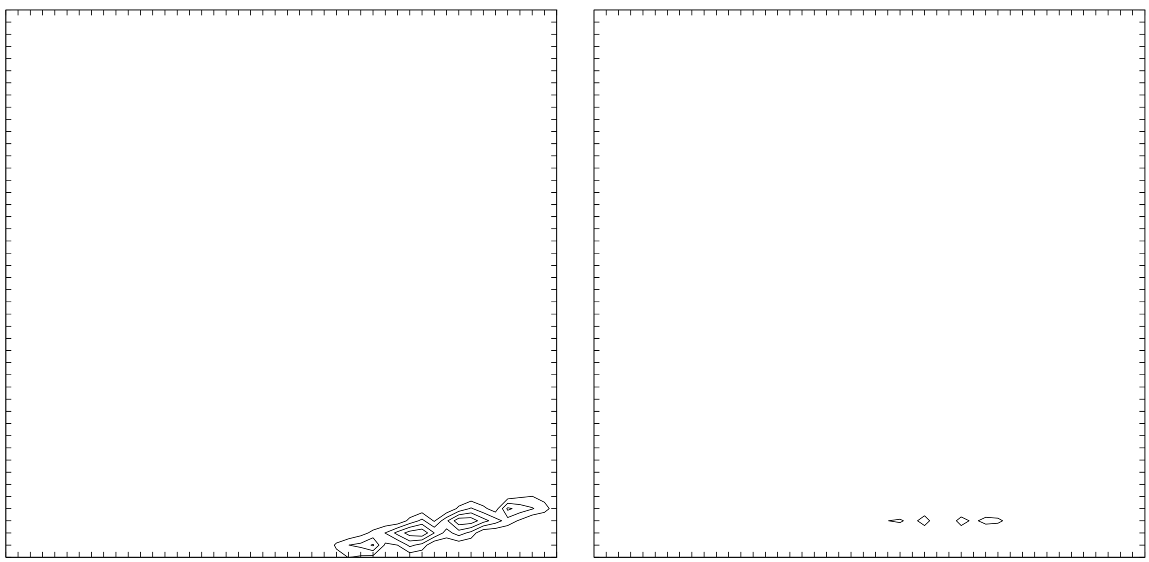

Figure 21: Close-ups of $\Gamma^{3,2}(m,k)$ et $\Gamma^{3,3}(m,k)$ for Coifman 12

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