ACCELERATED SPATIO-TEMPORAL WAVELET TRANSFORMS: AN ITERATIVE TRAJECTORY ESTIMATION. *

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ABSTRACT

This paper addresses the problem of estimating and analyzing accelerated motion in spatio-temporal discrete signals. It is assumed that the digital signals of interest are acquired from imaging sensors and structured as digital image sequences. The motion trajectories in the signal are two-dimensional spatial projections in time of three-dimensional motions. Consequently, they contain all the orders of acceleration. The purpose of this work is to estimate the trajectory and the motion parameters of selected moving objects in the scene. The final goal is to provide selective reconstructions of accelerated objects of interest. This paper presents the construction of new continuous wavelet transforms that can be tuned to any order of accelerations, we demonstrate their existence and provide the related admissibility conditions. The parameters for analysis that are taken into account in these accelerated wavelet transforms are spatial and temporal translations, velocity, acceleration (second or nth order), spatial scale and spatial rotation. The continuous wavelet functions are finally discretized for signal processing.

1. INTRODUCTION

In this paper, we demonstrate the existence of spatio-temporal continuous wavelet transforms that are built on acceleration parameters. The whole technique of calculation is outlined and the admissibility conditions are presented. This method leads to an iterative scheme of estimating trajectory parameters. These wavelet transforms extend our previous work [4] [7] [8] [9] done on the Galilean wavelets which were dedicated to analyzing spatial and temporal translations, velocity, rotation and scale. These new accelerated wavelet transforms take in to account the acceleration as an additional parameter. The uncertainty relations between some of these parameter especially between velocity and translation will be clearly evidenced from the group extension. The purpose of these continuous wavelets is to be discretized to analyze digital signals acquired by imaging sensors or radars.

The approach considered in this paper differs fundamentally from other techniques that have been proposed in the literature such as those based on optical flow, pel-recursive, block matching and Bayesian models. The continuous wavelet transform provides motion estimations that are robust not only against image noise and blur but also against motion noise (i.e. jitter). Moreover, as a result of both the spatio-temporal filtering and the temporal interpolation properties of continuous wavelet transforms, this technique can resolve temporary occlusion problems that may occur in the scene [7] [8] [9]. It has also been demonstrated that the wavelet transform behaves as a matched filter and performs minimum-mean-square error estimation of the motion parameters.

In this paper, we demonstrate first the existence of continuous wavelet transforms based on acceleration. The construction is based on the Lie group theory and their representations. The set of parameters is large and the group structure becomes quite complicated. The group and its law of composition state the analysis rules that apply in the signal when composing parameters. They have to comply to the dynamics embedded into the signal. The wavelet representation is a mapping (a homomorphism) between the elements of the group and the operator in the spatio-temporal Hilbert space H of the signal. To derive the existence of wavelet transforms, the representation must be unitary, irreducible and square integrable. In this paper, we will construct and induce unitary, and irreducible representations in the spatio-temporal Hilbert space and demonstrate that the representation is square integrable. The paper will show the existence of a central extension for the group and explain that there are several admissible kinematics corresponding to admissible Lie algebras. The central extension is an additional dimension that shows up in the space of the signal parameters and creates uncertainties. Finally, we present numerical results on synthetic and actual digital image sequences for the estimation of the acceleration.

2. TRAJECTORY AND ACCELERATION TRANSFORMATION

The trajectory $\vec{x}(t)$ of an object moving in a scene may be represented by a Taylor expansion at any point $(\vec{x}_0, \tau)$

$$\vec{x}(t=\tau) = \vec{x}_0 + \vec{v}\tau + \vec{\gamma}_0 \frac{\tau^2}{2!} + \sum_{i=1}^{\infty} \vec{\gamma}_i \frac{\tau^{i+2}}{(i+2)!} \qquad (1)$$

where $\vec{v}$ is the velocity and the $\vec{\gamma}_i$ are the accelerations of order i, $i \in Z_+$. The determination of the trajectory consists then in estimating the parameters of the Taylor expansion i.e. location, velocity and accelerations in increasing order as long as those parameters are significantly different from zero. In this paper, we will focus on $(\vec{b} = \vec{x}_0, \tau, \vec{v}, \vec{\gamma}_0)$. Moreover, we will add two other parameters, the scale a and the object orientation θ in space. The continuous wavelet transforms that tune only to velocity are called Galilean wavelets they have already been studied in [4]. Now, we proceed one step further by incorporating the acceleration.

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The motion transformation of the spatio-temporal referentials $(\vec{x}, t)$ that refers to the set of parameters of interest with acceleration $\vec{\gamma}_0$ can be written as follows

$$\vec{x}' = ra\vec{x} + \vec{b} + \vec{v}t + \frac{1}{2}\vec{\gamma}_0 t^2 \quad (2)$$

$$t' = t + \tau \quad (3)$$

where r, a, $\vec{b}$, $\vec{v}$, $\vec{\gamma}_0$ and τ stand for spatial rotation, scale, spatial translation, velocity, acceleration, temporal translation. Let us remark that $\vec{b} \in \mathbf{R}^2$, $\vec{v} \in \mathbf{R}^2$, $a \in \mathbf{R}^+\backslash\{0\}$, $\tau \in \mathbf{R}$, $r \in SO(2)$ where we assume two dimensions in space and one in time. By Ado theorem, any Lie group has a matrix representation. In this case, the transformation fits in

$$\begin{pmatrix} \vec{x}' \\ t' \\ t'^2 \\ 1 \end{pmatrix} = \begin{pmatrix} ar & \vec{v} & \vec{\gamma}_0/2 & \vec{b} \\ 0 & 1 & 0 & \tau \\ 0 & 2\tau & 1 & \tau^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{x} \\ t \\ t^2 \\ 1 \end{pmatrix} \quad (4)$$

The matrix multiplication provides the law of composition for the group that satisfies all the group postulates. Moreover, it is quite interesting to consider $t^2 = \kappa$ as an independent dimension in order to proceed in the calculations. At the end, we may always state $\kappa = t^2$. Then τ^2 becomes an additional parameter to be taken into account as the translation on the $\kappa = t^2$ dimension. Then, the third relation that reads as

$$\kappa' = \kappa + 2t\tau + \tau^2 \quad (5)$$

is to be added to 2 and 3.

3. LIE GROUP FOR ACCELERATION

In this section, we refer to the signal transformations in 2 3 and 5 to define the Lie group for acceleration. The group is defined by an element g, the law of composition, the inverse element or the identity. In this case, the group element is $g = \{\vec{b}, \tau, \tau^2, \vec{\gamma}_0, \vec{v}, a, r\}$. The law of composition and the inverse are then

$$\begin{aligned} g \circ g' &= \Big(\vec{b} + ar\vec{b}' + \vec{v}\tau' + \tfrac{1}{2}\vec{\gamma}_0\tau'^2,\ \tau + \tau'; \\ &\tau^2 + \tau'^2 + 2\tau\tau', aa',\ rr',\ \vec{v} + ar\vec{v}' + \vec{\gamma}_0\tau',\ \vec{\gamma}_0 + ar\vec{\gamma}_0' \Big) \\ (\vec{b}, \tau, \tau^2, \vec{v}, a, r, \vec{\gamma}_0)^{-1} &= \big(-a^{-1}r^{-1}[\vec{b} + \tfrac{1}{2}\vec{\gamma}_0 - \vec{v}\tau], \\ &-\tau, \tau^2, -a^{-1}r^{-1}[\vec{v} - \vec{\gamma}_0\tau];\ a^{-1},\ r^{-1},\ -a^{-1}r^{-1}\vec{\gamma}_0 \big) \end{aligned} \quad (6)$$

The study of the Lie algebra shows the existence of a one-dimensional central extension. Let ϕ be that parameter of the extension, the element of the extended group reads then as $\tilde{g} = \{\phi, g\}$. The structure of the group with the central extension yields

$$\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^2 \wedge \mathbf{R}^2 \wedge \mathbf{R}^2 \wedge \mathbf{R}_*^+ \times SO(2) \quad (7)$$

where $\times$ and $\wedge$ stand for direct and semi-direct products respectively and $\mathbf{R}_*^+ = \mathbf{R}^+\backslash\{0\}$.

4. LIE ALGEBRAS FOR ACCELERATION

The Lie algebra $\mathcal{G}$ is obtained from the Lie group G by derivation. It stands as the vector space tangent at the identity element for the group. The Lie algebra $\mathcal{G}$ can be represented as a Lie algebra of matrices. Each parameter in the Lie group corresponds in the Lie algebra to one matrix (called a generator). Let us define $\mathcal{G}_{op}$ the set of generators as follows: J, P, K, Q, H^1, H^2, S for rotation, scale, spatial translation, velocity, acceleration, temporal translation, square temporal translation and scale. An element ζ of the Lie algebra $\mathcal{G}$ with coordinates $(j, p, k, q, h_1, h_2, s)$ writes

$$\zeta = jJ + pP + kK + qQ + h_1H^1 + h_2H^2 + sS \quad (8)$$

The structure of the group G and the algebra $\mathcal{G}$ is determined by the set of all the commutation relationships of the form

$$[A\ B] = AB - BA \quad (9)$$

where A and B stands as any two Lie algebra operators. The calculation of the set of the commutators from 4 yields

$[S\ Q_1] = Q_1$; $[S\ Q_2] = Q_2$; $[S\ P_1] = P_1$; $[S\ P_2] = P_2$;
$[S\ K_1] = K_1$; $[S\ K_2] = K_2$; $[J\ Q_1] = Q_2$; $[J\ Q_2] = -Q_1$;
$[J\ P_1] = P_2$; $[J\ P_2] = -P_1$; $[J\ K_1] = K_2$; $[J\ K_2] = -K_1$;
$[J\ H^1] = 0$; $[J\ H^2] = 0$; $[K_i\ H^1] = P_i$; $[K_i\ H^2] = -K_i$;
$[K_i\ P_j] = 0$; $[K_i\ Q_j] = 0$; $[P_i\ Q_j] = 0$; $[P_i\ H^1] = 2K_i$;
$[P_i\ H^2] = P_i$; $[K_i\ H^1] = K_i$; $[Q_i\ H^2] = \frac{P_i}{2}$; $[H^1\ H^2] = 0$
where $i, j \in \{1, 2\}$ stand for the two spatial dimensions.

The operators must also fulfill the Jacobi relationships. These apply among any set of three operators A, B and C

$$[[A\ B]\ C] + [[B\ C]\ A] + [[C\ A]\ B] = 0 \quad (10)$$

Fulfilling all the Jacobi equations should be possible with several additional sets of commutators, i.e. different Lie algebra structures that define each an admissible kinematic (this entire search remains as further work with a computer routine). In this section, we focus only on the non-extended algebra (i.e. the previous set) and two extensions, namely a central and a non-central. The algebras of the central extension of the group is found by removing the scale operator and then it is easy to see that the commutators $[P_i\ K_j]$ may be defined up to a constant m in the Jacobi equation involving the generators $J - K - P$; i.e. $[P_i\ K_j] = \delta_{ij}mI$. m is the real variable of structure for the central extension of the group and I the corresponding generator. By definition, the generator I commutes with all the other generators of the algebra. Similarly to the Galilean case, the extension is one-dimensional since eventually $[K_i\ Q_j] = 0$, and $[P_i\ Q_j] = \frac{1}{2}\delta_{ij}mI$. The second step consists in introducing the scale operator S that defines a non-central extension of the Group (i.e. a subgroup S of the group of all the automorphisms of the normal subgroup N in $N \wedge S$).

Since the group is simply connected, the law of composition of the parameter ϕ for the central extension may be obtained by exponentiation of the algebra. The calculation proceeds by iteratively applying the Baker-Hausdorff formula

$$\exp[-A]B\exp[A] = A + [A\ B] + \frac{1}{2}[[A\ B]\ B] + \ldots \quad (11)$$

on the exponential representation of a group element

$$\tilde{g} \circ \tilde{g}' = \exp[\phi\ I]\exp[\tau\ H^1]\exp[\tau^2\ H^2]\exp[\vec{v}\ K]\exp[\vec{b}\ P]\ exp[\vec{\gamma}_0\ Q]\ exp[a\ S]\ R\exp[\phi'\ I]\ \exp[\tau'\ H\ 1]\ \exp[\tau^{2'}\ H^2]\ \exp[\vec{v}'\ K]\ \exp[\vec{b}'\ P]\ exp[\vec{\gamma}_0'\ Q]\ exp[a'\ S]\ R' \tag{12}$$

finally the whole calculation leads to

$$\phi'' = \phi + a^2\phi' + ma\vec{v}r\vec{b}' + \frac{1}{2}m\vec{v}^2\tau' + \frac{1}{2}am\vec{\gamma}_0 r\vec{b}' + \frac{1}{2}m\vec{\gamma}_0^2\tau^{2'} \tag{13}$$

This provides the law of composition of the extension parameter in the group. The inverse element $\tilde{g}$ reads now

$$\tilde{g}^{-1} = \{-a^2\left(\phi + \frac{1}{2}\vec{v}\vec{b} + \frac{1}{2}\vec{\gamma}_0\vec{b}\right),\ g^{-1}\} \tag{14}$$

The use of the exponentiation may be done at any stage since the rotation subgroup $SO(2)$ guarantees that the group be simply connected.

5. INTERPRETATION OF THE CENTRAL EXTENSION

The central extension appears as a hidden parameter in the transformation or as a complementary dimension imbedded in the signal. The structural parameter of the group extension m carries an interpretation in terms of spatio-temporal signal processing. m generates the uncertainty that holds between velocity ($\vec{v}$, operator K) and location (parameter $\vec{b}$, operator P). The uncertainty is currently observable when processing the signal with spatio-temporal wavelet transforms [9]. When any two operators like K and P fail to commute, the related quantity $\vec{v}$ and $\vec{b}$ can not be accurately measured simultaneously. The degree of the inevitable lack of precision is measured by their commutator $[K_i\ P_i] = K_iP_i - P_iK_i = m$. A Heisenberg relation reads as

$$< \Delta K >< \Delta P > \geq \frac{1}{2}\,|m| \tag{15}$$

where $< \Delta K >$ and $< \Delta P >$ stand for the uncertainty between $\vec{v}$ and $\vec{b}$. They are defined as the standard deviation of operators K and V i.e. the square root of their variance. This reasoning holds whenever any two operators fail to commute in the algebra. We can see from the previous section that there are many uncertainties embedded in the signal. Any set of admissible commutators define one particular algebra, one particular group and one particular kinematic.

6. ACCELERATED WAVELET TRANSFORMS

The usual way to derive a representation which is unitary and irreducible in the spatio-temporal Hilbert space is to consider the adjoint and the coadjoint actions of the Lie algebra. The adjoint action $ad[\tilde{g}]$ is given by the matrix of the conjugation at group identity

$$ad[\tilde{g}] = \frac{\partial(\tilde{g}\tilde{g}'\tilde{g}^{-1})}{\partial\tilde{g}'}|_{\tilde{g}'=e} \tag{16}$$

The coadjoint action is then given in the phase space as $coad[\tilde{g}^{-1}] = ad[\tilde{g}]^T$ where ^{T} stands for the transpose. The coadjoint action enables then to induce a group representation which is unitary and irreducible and reads as

$$[T(\tilde{g})\hat{\Psi}]\left(m,\vec{k},\omega,\sigma\right) = a^{1/2}e^{i(m\phi+\vec{k}\vec{b}+\omega\tau+\sigma\tau^2)}\hat{\Psi}\left(m',\vec{k}',\omega',\sigma'\right) \tag{17}$$

with

$$\begin{aligned} \vec{k}' &= ar^{-1}(k - m\vec{v} + \tfrac{1}{2}m\vec{\gamma}_0); & \omega' &= \omega + \vec{v}\vec{k} + \tfrac{1}{2}m\vec{v}^2 \\ \sigma' &= \sigma + \tfrac{1}{2}\vec{k}\vec{\gamma}_0 + \tfrac{1}{2}m\vec{\gamma}_0^2; & m' &= a^2 m \end{aligned} \tag{18}$$

As usual, the exponential carries the characters of the normal subgroup. m, $\vec{k}$, ω and σ belong to the dual space of the normal subgroup and then stand as Fourier variables. To define a wavelet transform, the representation has to additionally fulfill square integrability. Let us then define the measure on the group and integrate. The group is non-unimodular since the left and right Haar measure μ_l and μ_r are different (indeed $det|\frac{\partial(\tilde{g}\tilde{g}')}{\partial\tilde{g}'}|^{-1}_{\tilde{g}'=e} = a^{-(3n+1)}$ n-dimensional space, $det|\frac{\partial(\tilde{g}'\tilde{g})}{\partial\tilde{g}'}|^{-1}_{\tilde{g}'=e} = a^{-1}$) and given by

$$\begin{aligned} \mu_l &= a^{-(3n+1)}d\phi da d\vec{b} d\tau d\tau^2 d\vec{v} d\vec{\gamma}_0 dr \\ \mu_r &= a^{-1}d\phi da d\vec{b} d\tau d\tau^2 d\vec{v} d\vec{\gamma}_0 dr \end{aligned} \tag{19}$$

The condition of squared integrability is then verified on the group G (i.e. after sectioning at $\phi = \phi_0$) is then given as

$$\int_G d\mu_l |< T(g)\Psi|\eta >|^2 = c(\Psi,\eta) < \infty\,;\ \forall g \in G\,,\ \forall \eta \in H \tag{20}$$

After integrating on the translations and the rotations, the condition reduces to

$$\int \frac{da d\vec{v} d\vec{\gamma}_0}{a^{3n}}|\hat{\Psi}(\vec{k},\omega,\sigma)|^2 < \infty \tag{21}$$

In the case of the one-dimensional space and time, the condition reads

$$\int dk' d\omega' d\sigma' \frac{|\hat{\Psi}(k',\omega',\sigma')|^2}{|\vec{k}'|^3} < \infty \tag{22}$$

and there is square integrability for all $m \in \mathbf{R}$. The numerator was $|\vec{k}'|^2$ in the Galilean case, is now $|\vec{k}'|^3$ and generalizes to $|\vec{k}'|^{(n+3)}$ when considering $\vec{v}, \vec{\gamma}_0, \ldots, \vec{\gamma}_n$. In the case of the two-dimensional space and time, the condition depends also on the Jacobian of the transformation

$$\begin{aligned} \vec{k}' &= ar^{-1}(k - m\vec{v} + \tfrac{1}{2}m\vec{\gamma}_0); & \omega' &= \omega + \vec{v}\vec{k} + \tfrac{1}{2}m\vec{v}^2 \\ \sigma' &= \sigma + \tfrac{1}{2}\vec{k}\vec{\gamma}_0 + \tfrac{1}{2}m\vec{\gamma}_0^2; \end{aligned} \tag{23}$$

to be different from zero, the condition of admissibility needs $m \in \mathbf{R}\backslash\{0\}$ to be satisfied. In multidimensional spatial cases, $m \neq 0$ and uncertainty is unvoidable. The calculations may generalize for acceleration of Higher orders. The technique of estimation is iterative: first estimate location on the trajectory, then velocity at that location, then acceleration of first order at that location and velocity,....

7. MORLET WAVELET AND APPLICATIONS

The applications presented for the estimation of the acceleration have been performed with accelerated wavelets tuned to the first order of acceleration γ_0. An anisotropic *Morlet wavelet* is admissible as a mother wavelet in the accelerated family. The wavelet is first calculated in the space and time domain and then transformed to the Fourier domain where the inner product $< T(g)\Psi|S > |$ is computed. Figure 1 displays a synthetic scene with three objects in accelerated motion. A noise of 10 dBSNR is superposed on the scene and a jitter (Gaussian, 1 pel of standard deviation) is apply to the motion. Temporary crossings occur in the scene. Figure 2 shows the energy densities i.e. the square modulus of the wavelet transform in the plane of the accelerations $(\gamma_{0x}, \gamma_{0y})$. Figures 3 and 4 present the same results for a natural scene.

8. CONCLUSIONS

In this paper, we have demonstrated the existence of new spatio-temporal continuous wavelet transforms that are tuned to accelerated motion. Their admissibility conditions have been established. The technique of calculation has also been outlined. They generalizes the previous works that have been done for the affine and the Galilean wavelets respectively. Moreover, the wavelet construction that is proposed here generalizes itself to any order of acceleration (at the expense of even longer calculations) without changing fundamentally the principles and the main conclusions. Eventually, applications dealing with actual digital image sequences may be derived that show the usefulness of these wavelets to estimate motion parameters on trajectories. Further work to be presented will include the trajectory construction, the study of all the admissible and physical kinematics and some selective reconstructions of accelerated objects in natural scenes.

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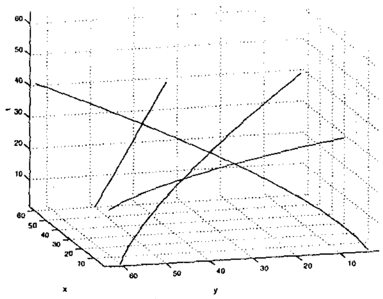

Figure 1. Synthetic scene with three accelerated objects, Gaussian noise 10dB SNR and motion jitter of 1 pixel standard deviation.

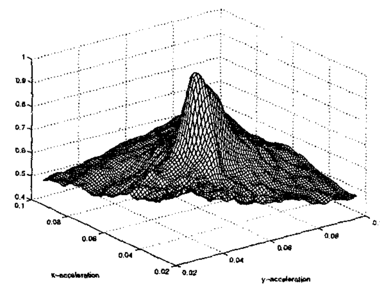

Figure 2. Detection of the acceleration on a trajectory ($\tau = 2$, $\vec{v} = \{0.5, 0.5\}$). The object is starting at $\vec{b} = \{3, 3\}$ and velocity $\vec{v} = \{0.5, 0.5\}$. The estimated acceleration is $\vec{\gamma} = \{0.05, 0.05\}$.

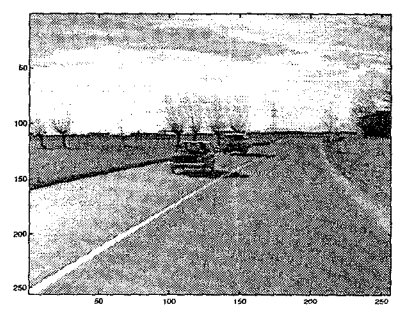

Figure 3. Actual image sequence with two cars approaching at constant velocity. Due to the projection on the sensor plane, the motion is virtually accelerated.

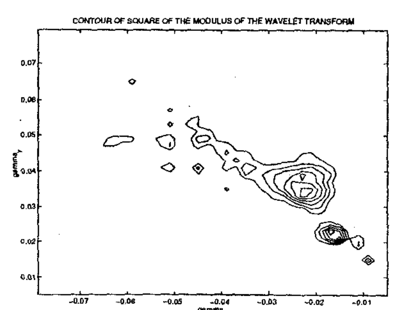

Figure 4. Two local maxima at $\vec{\gamma} = \{-0.022, +0.034\}$ and $\vec{\gamma} = \{-0.018, +0.0221\}$ computed in the previous scene in the plane of the image correspond to each car.