Discrete Wavelet Transforms in Practice

Wei ZHU and M. Victor WICKERHAUSER
Washington University in St. Louis, Missouri
victor@math.wustl.edu
http://www.math.wustl.edu/~victor

SPIE Conference 7343-02, Orlando, Florida
April 13th, 2009

Main Ideas

Goal: implement discrete wavelet transforms efficiently

- **Lifting:** reduce the number of arithmetic operations.
- **Nearest-neighbors:** reduce the number of memory accesses.
- **Symmetry:** simplify the implementation.
- **Choice:** maximize some utility.

Starting point: Sweldens' and Daubechies' lifting factorization.

Discrete Wavelet Transforms

Signal: $u \in \ell^2$, in practice finitely supported or periodic.

Analysis filters: linear maps $\tilde{H}, \tilde{G} : \ell^2 \to \ell^2$; convolution and subsampling.

Discrete wavelet transform:

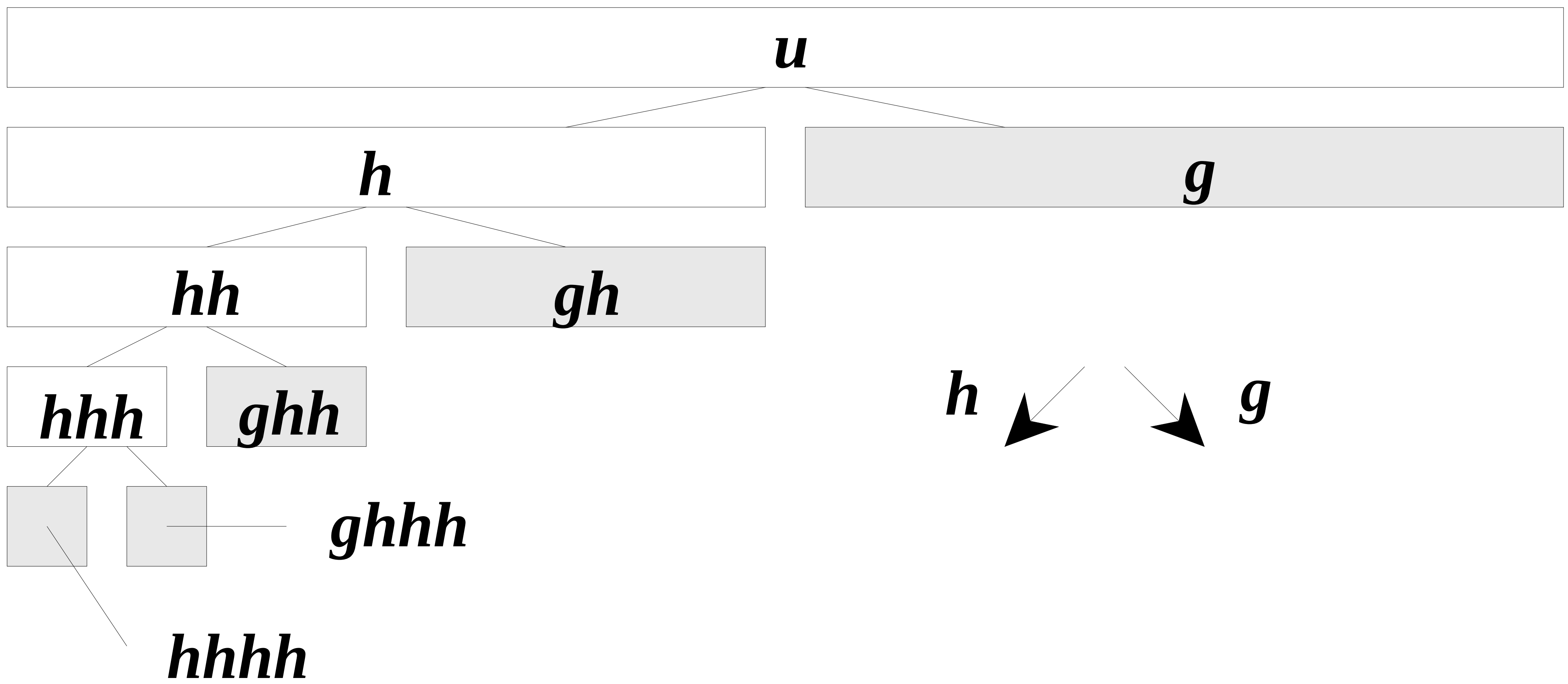
$$u \mapsto \{\tilde{H}^J u;\ \tilde{G}\tilde{H}^{J-1}u,\ \tilde{G}\tilde{H}^{J-2}u,\ \ldots,\ \tilde{G}\tilde{H}u,\ \tilde{G}u\}.$$

Synthesis filters: linear maps $H, G : \ell^2 \to \ell^2$, related to $\tilde{H}, \tilde{G}$.

Wavelet reconstruction:

$$\begin{aligned} u &= G^*\tilde{G}u + H^*\tilde{H}u \\ &= G^*\tilde{G}u + H^*\left(G^*\tilde{G}\tilde{H}u + H^*\tilde{H}^2u\right) \\ &= \cdots \\ &= G^*\tilde{G}u + H^*\left(G^*\tilde{G}\tilde{H}u + H^*\left(\cdots + H^*\left(G^*\tilde{G}\tilde{H}^{J-1}u + H^*\tilde{H}^J u\right)\right)\cdots\right). \end{aligned}$$

Discrete Wavelet Transforms II



Use adjoints h^*, g^* to reconstruct u, moving up and adding.

Filters, Adjoints, and Conjugates

A *(finite) filter* $F : \ell^2 \to \ell^2$ is a linear transformation determined by a (finitely-supported) absolutely summable sequence $f = \{f_n : n \in \mathbf{Z}\}$:

$$Fx_m = \sum_n f_{2m-n} x_n, \qquad m \in \mathbf{Z}.$$

The *adjoint filter* F^* determined by the same sequence f is

$$F^* x_n = \sum_m \bar{f}_{2m-n} x_m, \qquad n \in \mathbf{Z}.$$

Thus $\langle Fx, y\rangle = \langle x, F^* y\rangle$ for all $x, y \in \ell^2$, using the inner product in ℓ^2.

The *conjugate filter* $\dot{F}$ of F has sequence $\dot{f} = \{\dot{f}_n : n \in \mathbf{Z}\}$ defined by

$$\dot{f}_n = (-1)^n f_{1-n} \qquad \Rightarrow \quad f_n = (-1)^{1-n} \dot{f}_{1-n}, \quad \Rightarrow \ddot{F} = -F$$

If F is finite then $\dot{F}$ is also finite, with the same support length.

Orthogonality, Biorthogonality, and Perfect Reconstruction

Filter H is called *orthogonal* if it and its conjugate filter $G = \dot{H}$ satisfy

$$HH^* = Id; \quad GG^* = Id; \quad GH^* = HG^* = 0; \quad H^*H + G^*G = Id.$$

Filters H, G form a *perfect reconstruction pair* if they and their conjugates $\tilde{H} = \dot{G}$ and $\tilde{G} = \dot{H}$ satisfy

$$\tilde{H}H^* = Id; \quad \tilde{G}G^* = Id; \quad \tilde{G}H^* + \tilde{H}G^* = 0; \quad H^*\tilde{H} + G^*\tilde{G} = Id.$$

These are also called *biorthogonality conditions*. Since $\ddot{H} = -H$ and $\ddot{G} = -G$,

$$H\tilde{H}^* = Id; \quad G\tilde{G}^* = Id; \quad G\tilde{H}^* + H\tilde{G}^* = 0; \quad \tilde{H}^*H + \tilde{G}^*G = Id.$$

Thus $(\tilde{H}, \tilde{G}) = (\dot{G}, \dot{H})$ form a perfect reconstruction pair as well.

EG: If H is an orthogonal filter, then $(H, \dot{H})$ is a perfect reconstruction pair.

Call H a *perfect reconstruction filter* if there is a filter G such that (H, G) is a perfect reconstruction filter pair.

Perfect Reconstruction Conditions for Sequences

Suppose $H \leftrightarrow h$ has conjugate $G \leftrightarrow g$.

Orthogonality:

$$\sum_k h(k)\bar{h}(k+2n) \;=\; \mathbf{1}(n) \;=\; \sum_k g(k)\bar{g}(k+2n);$$

$$\sum_k g(k)\bar{h}(k+2n) \;=\; 0 \;=\; \sum_k h(k)\bar{g}(k+2n);$$

and

$$\sum_k h(2k+m)\bar{h}(2k+n) + \sum_k g(2k+m)\bar{g}(2k+n) = \mathbf{1}(n-m).$$

Here

$$\mathbf{1}(n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases} \qquad n, m \in \mathbf{Z}.$$

Exercise: find the conditions for biorthogonal perfect reconstruction pairs.

Z Transforms

Z *transform* of a sequence $x = \{x_n \in \mathbf{C} : n \in \mathbf{Z}\} \in \ell^2$:

$$x(z) = \sum_n x_n z^{-n}, \qquad \text{with} \qquad \begin{array}{ll} \textit{even part} & x_e(z) \stackrel{\text{def}}{=} \sum_n x_{2n} z^{-n}; \\ \textit{odd part} & x_o(z) \stackrel{\text{def}}{=} \sum_n x_{2n+1} z^{-n}. \end{array}$$

Recover the Z-transform of x from the even and odd parts x_e, x_o:

$$x(z) = x_e(z^2) + z^{-1} x_o(z^2).$$

Get the even and odd parts from the Z-transform:

$$x_e(z^2) = \frac{x(z) + x(-z)}{2}, \qquad x_o(z^2) = \frac{x(z) - x(-z)}{2z^{-1}}.$$

Laurent Polynomials

If $p \in \ell^2$ is finitely-supported, then its Z transform $p(z)$ is a *Laurent polynomial*:

$$p(z) = \sum_{n=a}^{b} p_n z^{-n}, \qquad a \le b, \quad a, b \in \mathbf{Z}.$$

If $p \not\equiv 0$, define its *degree* by $\deg p = b - a$.

Laurent polynomials form the commutative ring $\mathbf{C}[z, z^{-1}]$ with multiplicative identity 1.

Element $p \not\equiv 0$ is called a *unit*

$\iff$ p has a multiplicative inverse

$\iff$ p is a *monomial* $p(z) = Kz^n$

$\iff$ $\deg p = 0$.

Then $p^{-1}(z) = K^{-1} z^{-n}$.

Conjugates and Z Transforms

If $\dot{F}$ is the conjugate of filter F, then

$$\dot{f}(z) = -z^{-1} f(-z^{-1}), \qquad \begin{aligned} \dot{f}_e(z) &= f_o(z^{-1}), \\ \dot{f}_o(z) &= -f_e(z^{-1}). \end{aligned}$$

Remark: it is possible to generalize to the M-*conjugate* for fixed $M \in \mathbf{Z}$:

$$\dot{f}_n = (-1)^n f_{2M+1-n} \qquad \Rightarrow \quad f_n = (-1)^{1-n} \dot{f}_{2M+1-n}, \quad \Rightarrow \ddot{F} = -F$$

For the M-conjugate of filter F, compute

$$\dot{f}(z) = -z^{-2M-1} f(-z^{-1}), \qquad \begin{aligned} \dot{f}_e(z) &= z^{-2M} f_o(z^{-1}), \\ \dot{f}_o(z) &= -z^{-2M} f_e(z^{-1}). \end{aligned}$$

Filter and Adjoint Filter Action on Z Transforms

$$
\begin{aligned}
Fx(z) &= \sum_m Fx_m z^{-m} = \sum_m \sum_n f_{2m-n} x_n z^{-m} \\
&= \sum_m \sum_n f_{2m-2n} x_{2n} z^{-m} + \sum_m \sum_n f_{2m-2n-1} x_{2n+1} z^{-m} \\
&= \left(\sum_m f_{2m} z^{-m}\right)\left(\sum_n x_{2n} z^{-n}\right) + \left(\sum_m f_{2m-1} z^{-m}\right)\left(\sum_n x_{2n+1} z^{-n}\right) \\
&= \left(\sum_m f_{2m} z^{-m}\right)\left(\sum_n x_{2n} z^{-n}\right) + z^{-1}\left(\sum_m f_{2m+1} z^{-m}\right)\left(\sum_n x_{2n+1} z^{-n}\right) \\
&= f_e(z)x_e(z) + z^{-1} f_o(z) x_o(z);
\end{aligned}
$$

$$
\begin{aligned}
F^*x(z) &= \sum_n F^*x_n z^{-n} = \sum_n \sum_m \bar{f}_{2m-n} x_m z^{-n} \\
&= \sum_m \sum_n \bar{f}_n x_m z^{-n-2m} = \left(\sum_n \bar{f}_n z^{-n}\right)\left(\sum_m x_m z^{-2m}\right) \\
&= \bar{f}(z) x(z^2),
\end{aligned}
$$

Filter and Adjoint Filter Action on Z Transforms II

Alternate "correlation" definition of (finite) filter and adjoint:

$$Fx_m = \sum_n f_{2m+n} x_n, \quad m \in \mathbf{Z}; \qquad F^* x_n = \sum_m \bar{f}_{2m+n} x_m, \quad n \in \mathbf{Z}.$$

$$\begin{aligned}
Fx(z) &= \sum_m Fx_m z^{-m} = \sum_m \sum_n f_{2m+n} x_n z^{-m} \\
&= \sum_m \sum_n f_{2m+2n} x_{2n} z^{-m} + \sum_m \sum_n f_{2m+2n+1} x_{2n+1} z^{-m} \\
&= \left(\sum_m f_{2m} z^{-m} \right) \left(\sum_n x_{2n} z^{-n} \right) + \left(\sum_m f_{2m+1} z^{-m} \right) \left(\sum_n x_{2n+1} z^{-n} \right) \\
&= f_e(z) x_e(z) + f_o(z) x_o(z),
\end{aligned}$$

$$\begin{aligned}
F^* x(z) &= \sum_n F^* x_n z^{-n} = \sum_n \sum_m \bar{f}_{2m+n} x_m z^{-n} \\
&= \sum_m \sum_n \bar{f}_n x_m z^{-n+2m} = \left(\sum_n \bar{f}_n z^{-n} \right) \left(\sum_m x_m z^{2m} \right) \\
&= \bar{f}(z) x(z^{-2}),
\end{aligned}$$

Perfect Reconstruction in Terms of Z Transforms

Perfect reconstruction conditions for filters $H, G, \tilde{H} = \dot{G}, \tilde{G} = \dot{H}$, in terms of their Z transforms:

$$h(z)\tilde{h}(z^{-1}) + g(z)\tilde{g}(z^{-1}) = 1; \qquad h(z)\tilde{h}(-z^{-1}) + g(z)\tilde{g}(-z^{-1}) = 0.$$

In terms of the even and odd parts:

$$h_e(z)\tilde{h}_e(z^{-1}) + g_e(z)\tilde{g}_e(z^{-1}) = 1; \quad h_e(z)\tilde{h}_o(z^{-1}) + g_e(z)\tilde{g}_o(z^{-1}) = 0;$$

$$h_o(z)\tilde{h}_o(z^{-1}) + g_o(z)\tilde{g}_o(z^{-1}) = 1; \quad h_o(z)\tilde{h}_e(z^{-1}) + g_o(z)\tilde{g}_e(z^{-1}) = 0$$

The Polyphase Representation

The *polyphase matrix* of a pair H, G of finite filters and their conjugates $\tilde{H} = \dot{G}$, $\tilde{G} = \dot{H}$, is

$$P(z) = \begin{bmatrix} h_e(z) & g_e(z) \\ h_o(z) & g_o(z) \end{bmatrix}, \qquad \tilde{P}(z) = \begin{bmatrix} \tilde{h}_e(z) & \tilde{g}_e(z) \\ \tilde{h}_o(z) & \tilde{g}_o(z) \end{bmatrix}$$

P and $\tilde{P}$ belong to $\mathbf{Mat}(2 \times 2, \mathbf{C}[z, z^{-1}])$, the ring of 2×2 matrices over the Laurent polynomials.

The perfect reconstruction condition for even and odd parts is equivalent to:

$$P(z)\tilde{P}(z^{-1})^t = Id.$$

In practice, Id may be replaced by any diagonal matrix that is invertible in $\mathbf{Mat}(2 \times 2, \mathbf{C}[z, z^{-1}])$, the 2×2 matrices over the Laurent polynomials.

Matrices of Laurent Polynomials

Matrix ring $\mathbf{Mat}\left(2 \times 2, \mathbf{C}[z, z^{-1}]\right)$ has elements:

$$M(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}, \qquad a, b, c, d \in \mathbf{C}[z, z^{-1}].$$

M is invertible iff $\det M \in \mathbf{C}[z, z^{-1}]$ is invertible, namely is a monomial Kz^n. Then

$$M^{-1}(z) = K^{-1} z^{-n} \begin{bmatrix} d(z) & -b(z) \\ -c(z) & a(z) \end{bmatrix}.$$

Say that a Laurent polynomial $h(z)$ has a *complement* $g = g(z)$ if their polyphase matrix is invertible.

Finite filter H is part of a perfect reconstruction pair iff its Z-transform has a complement. This reduces part of filter design to algebra.

Division for Laurent Polynomials

$\mathbf{C}[z, z^{-1}]$ is a Euclidean domain: for $a, b \in \mathbf{C}[z, z^{-1}]$ with $\deg a \geq \deg b \geq 0$,

Lemma 1 *There exists a* quotient q *and a* remainder r *with* $\deg r < \deg b$ *so that*

$$a(z) = q(z)b(z) + r(z).$$

Note that $\deg q = \deg a - \deg b$.

Write $q = a/b$ and $r = a\%b$, as in C, but note that neither q nor r are unique.

Lemma 2 *There are at most* $2^{1+\deg a - \deg b}$ *different ways to divide* $a(z)/b(z)$*, among which at most* $2 + \deg a - \deg b$ *quotients are different.*

Example: Let $a(z) = 2z^{-1} + 4 + z$ and $b(z) = 1 + z$, then

$$\begin{aligned} a(z) &= (2z^{-1} + 1)b(z) + 1 && \text{(symmetric division)} \\ a(z) &= (3z^{-1} + 1)b(z) + (-z^{-1}) && \text{(right division)} \\ a(z) &= (2z^{-1} + 2)b(z) + (-z) && \text{(left division)} \end{aligned}$$

Note: division is a generalization of Gaussian elimination.

Greatest Common Divisors for Laurent Polynomials

Write $b|a$ (b *divides* a) if $a = qb + 0$ for some q. Thus $b|a \Rightarrow \deg b \leq \deg a$.

Say that d is a *common divisor* of a and b if $d|a$ and $d|b$.

Say that a common divisor d is a *greatest common divisor* of a and b if every common divisor c of a and b also divides d.

Lemma 3 *If d_1 and d_2 are greatest common divisors for a and b, then $d_1 = ud_2$ for some unit $u \in \mathbf{C}[z, z^{-1}]$.*

Theorem 4 *Every pair $a, b \in \mathbf{C}[z, z^{-1}]$, not both zero, has a greatest common divisor that is unique up to multiplication by a unit.*

Denote this set of greatest common divisors by $\gcd(a, b)$.

Say that a, b are *coprime* if $\gcd(a, b)$ is contained in the set of units.

Euclidean Algorithm for Laurent Polynomials

Assume a, b are Laurent polynomials with $\deg a \geq \deg b \geq 0$.

Put $a_0 \overset{\text{def}}{=} a$ and $b_0 \overset{\text{def}}{=} b$, and define a_k, b_k recursively:

$$a_{k+1} = b_k; \qquad b_{k+1} = a_k - q_k b_k, \qquad\qquad k = 0, 1, 2, \ldots,$$

where q_k is one of the possible quotients a_k/b_k. It thus determines b_{k+1} as the corresponding one of the possible remainders $a_k \% b_k$.

Lemma 5 *Let n be the smallest positive integer for which $b_n = 0$. Then $a_n \in \text{gcd}(a, b)$.*

Note that a_n is determined only up to a unit, defined by the sequence of quotients $q_0, \ldots, q_{n-1}$.

Theorem 6 *Laurent polynomial h has a complement g if and only if h_e and h_o are coprime.*

Proof: apply the Euclidean Algorithm to find the polyphase matrix.

The Euclidean Algorithm as Matrix Factorization

Write the recursion in matrix form:

$$\begin{bmatrix} a_{k+1}(z) \\ b_{k+1}(z) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -q_k(z) \end{bmatrix} \begin{bmatrix} a_k(z) \\ b_k(z) \end{bmatrix}, \quad \Rightarrow \begin{bmatrix} a_n(z) \\ 0 \end{bmatrix} = \prod_{k=1}^{n} \begin{bmatrix} 0 & 1 \\ 1 & -q_{n-k}(z) \end{bmatrix} \begin{bmatrix} a(z) \\ b(z) \end{bmatrix}.$$

Inverting the product of matrices gives

$$\begin{bmatrix} a(z) \\ b(z) \end{bmatrix} = (-1)^n \prod_{k=0}^{n-1} \begin{bmatrix} q_k(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n(z) \\ 0 \end{bmatrix}$$

If n is odd, absorb the unit $(-1)^n$ term into a_n.

Put $a = h_e$ and $b = h_o$ and assume gcd$(h_e, h_o) = Kz^m$, $K \neq 0$. Define g_e, g_o by

$$P(z) \stackrel{\text{def}}{=} \begin{bmatrix} h_e(z) & g_e(z) \\ h_o(z) & g_o(z) \end{bmatrix} = \prod_{k=0}^{n-1} \begin{bmatrix} q_k(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Kz^m & 0 \\ 0 & K^{-1}z^{-m} \end{bmatrix}.$$

Then $P(z)$ is evidently invertible. Get $\tilde{h}, \tilde{g}$ from $\tilde{P}(z^{-1})^t = P(z)^{-1}$.

Factorization into Lifting Steps

Theorem 7 (Daubechies and Sweldens) *For every perfect reconstruction finite filter pair* (H, G) *with polyphase matrix* P*, there exist finitely many Laurent polynomials* $s_i(z)$ *and* $t_i(z)$*,* $1 \le i \le m < \infty$*, and a non-zero constant* K *such that*

$$P(z) = \prod_{i=1}^{m} \begin{bmatrix} 1 & s_i(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_i(z) & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & 1/K \end{bmatrix}.$$

Unit upper triangular is called *Prediction*: $u_e \leftarrow u_e + S u_o$.

Unit lower triangular is called *Updating*: $u_o \leftarrow u_o + T u_e$.

Last diagonal matrix is called *Scaling*: $u_e \leftarrow K u_e$, $u_o \leftarrow K^{-1} u_o$.

Since u_e, u_e may be stored as disjoint arrays, this transform can be performed in place, without extra memory for temporary results.

Convert Factors into Lifting Steps

Proof through a series of observations. First:

$$\begin{bmatrix} q_k(z) & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & q_k(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_k(z) & 1 \end{bmatrix}.$$

The *flip* matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ cancel if Predict and Update steps alternate.

Factor a leftover flip matrix into lifting steps in a number of ways:

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\end{aligned}$$

Factoring Shift Matrices Into Lifting Steps

$$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1-z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1-z & 1 \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix}$$

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1+z & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Other factorizations exist, but at most 5 lifting steps are needed per shift.

Factor $\begin{bmatrix} z^{m} & 0 \\ 0 & z^{-m} \end{bmatrix}$ or $\begin{bmatrix} z^{-m} & 0 \\ 0 & z^{m} \end{bmatrix}$ into $5m$ lifting steps

Nearest Neighbor Factorization

The lifting factorization

$$P(z) = \prod_{k=1}^{n} \begin{bmatrix} 1 & s_k(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_k(z) & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & 1/K \end{bmatrix}.$$

uses only *nearest neighbors* if it satisfies the following conditions:

$$\begin{aligned} s_k(z) &= \alpha_k + \beta_k z^{-1}, \\ t_k(z) &= \gamma_k z + \delta_k, \end{aligned}$$

with $\alpha_k, \beta_k, \gamma_k, \delta_k \in \mathbf{C}$.

Nearest neighbor predict step: $u_{2k} \leftarrow u_{2k} + \alpha_k u_{2k-1} + \beta_k u_{2k+1}$.

Nearest neighbor update step: $u_{2k+1} \leftarrow u_{2k+1} + \gamma_k u_{2k} + \delta_k u_{2k+2}$.

PRO: Easy Symmetric Extension for Nearest Neighbors

Assume $u \in \ell^2$ is finitely supported in the index interval $[0, N-1]$.

Big $|u(0)|$ or $|u(N-1)|$ causes problems for filters.

Big $|u(N-1) - u(0)|$ causes problems for periodized filters.

If H, G are symmetric, use symmetric extension: Define

$$u(n) = \begin{cases} u(-n) & \text{, if } -N < n < 0; \\ u(n) = u(2N-1-n), & \text{if } N < n < 2N \end{cases}$$

and treat u as $2N-2$-periodic. Several other extensions are possible, depending on the symmetry type of H, G.

Symmetric extension nearest neighbor predict step:

$$u_{2k} \leftarrow u_{2k} + \begin{cases} \alpha(u_{2k-1} + u_{2k+1}), & \text{if } 2k \neq 0; \\ 2\alpha\, u_{2k+1}, & \text{if } 2k = 0. \end{cases}$$

Symmetric extension nearest neighbor update step:

$$u_{2k+1} \leftarrow u_{2k+1} + \begin{cases} \gamma(u_{2k} + u_{2k+2}), & \text{if } 2k+1 \neq N-1; \\ 2\gamma\, u_{N-2}, & \text{if } 2k+1 = N-1. \end{cases}$$

Example

Not all perfect reconstruction filters give nearest neighbor factorization directly. Let

$$h(z) = \frac{1}{\sqrt{2}}(1 + z^{-9}) \qquad g(z) = \frac{1}{\sqrt{2}}(-z^{8} + z^{-1}).$$

This is similar to the Haar orthogonal filter pair.

Then

$$h_e(z) = \tfrac{1}{\sqrt{2}} \qquad g_e(z) = \tfrac{1}{\sqrt{2}}z^4$$

$$h_o(z) = \tfrac{1}{\sqrt{2}}z^{-4} \quad g_o(z) = \tfrac{1}{\sqrt{2}}$$

The ordinary lifting factorization gives:

$$P(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & z^4 \\ z^{-4} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z^{-4} & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2}z^4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

This is not a nearest-neighbor factorization.

Factoring To Degree 1

Nearest neighbor lifting steps have off-diagonal terms $s(z), t(z)$ with $\deg s \leq 1$ and $\deg t \leq 1$.

Obtain this condition by decomposition:

$$\begin{bmatrix} 1 & s_1(z) + s_2(z) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & s_1(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s_2(z) \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ t_1(z) + t_2(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t_1(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_2(z) & 1 \end{bmatrix}.$$

But note that this may create multiple successive predict factors and multiple successive update factors.

Partial Long Division

Obtain degree 1 lifting factors at the Euclidean algorithm stage:

Theorem 8 *Assume a and b are two coprime nonzero Laurent polynomials. Then there exist Laurent polynomials $q_1, ..., q_n$ with $\deg q_k \leq 1$ for all $k = 1, ..., n$, such that*

$$\begin{bmatrix} a(z) \\ b(z) \end{bmatrix} = \prod_{k=1}^{n} \begin{bmatrix} q_k(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix},$$

where $n \leq 2(\deg a + \deg b + 1)$.

But note that the degree condition is not enough to guarantee that the factorization gives a nearest neighbor filter transform.

From Degree 1 to Nearest Neighbor

$$\begin{bmatrix} 1 & z^{2m}(\alpha z^{-1} + \beta) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z^m & 0 \\ 0 & z^{-m} \end{bmatrix} \begin{bmatrix} 1 & \alpha z^{-1} + \beta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-m} & 0 \\ 0 & z^m \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ z^{2m}(\gamma + \delta z) & 1 \end{bmatrix} = \begin{bmatrix} z^{-m} & 0 \\ 0 & z^m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \gamma + \delta z & 1 \end{bmatrix} \begin{bmatrix} z^m & 0 \\ 0 & z^{-m} \end{bmatrix},$$

where m is any integer and $\alpha, \beta, \gamma, \delta$ are constants.

Then factor the z^m shifts into at most 5m nearest neighbor lifting steps.

Condition Number for Filter Bank

If $P(z)$ is the polyphase matrix of a perfect reconstruction filter pair, then

$$\operatorname{cond}(P) \stackrel{\text{def}}{=} \frac{\sup\left\{\sqrt{\lambda_{\max}(P(z)^* P(z))} : |z| = 1\right\}}{\inf\left\{\sqrt{\lambda_{\min}(P(z)^* P(z))} : |z| = 1\right\}}.$$

If $P = P_1 \cdots P_n$, then $\operatorname{cond}(P) \leq \operatorname{cond}(P_1) \cdots \operatorname{cond}(P_n)$.

Assume the worst case (equality) and estimate the condition number for three implementations of the filter bank:

1. the original polyphase matrix P,

2. the usual (shortest) lifting factorization of P, and

3. the nearest-neighbor lifting factorization of P.

CON: Condition Numbers for Nearest Neighbor Factorization

Filter	Cond of P(z)	Cond of Lifting	Cond of N-N
9-7	1.32	205	205
D4	1	77	77
D6	1	76	76
Cubic B-spline	4	56	56
CDF-1-1	1	8.59	8.59
CDF-1-3	1.28	8.72	3100
CDF-1-5	1.42	6.25	1200
CDF-2-2	2	8.59	8.59
CDF-2-4	2	99	1900
CDF-3-1	4	643	643
CDF-3-3	4	723	3200
CDF-4-2	8	111	111
CDF-4-4	8	113	2800

PRO: Choose an Optimizing Lifting Factorization

Choose the lifting steps to obtain a nearest neighbor algorithm.

Choose the sequence of quotients $\{q_k(z) : k = 0, 1, \ldots, n-1\}$ to minimize condition number.

Choose the filter indexing to minimize the number of z^m shift matrices.

References

- C. Brislawn. “Classification of Nonexpansive Symmetric Extension Transforms for Multirate Filter Banks.” *Applied and Computational Harmonic Analysis*, 3:4(1996),337–357.

- I. Daubechies and W. Sweldens. “Factoring Wavelet Transforms into Lifting Steps.” *Fourier Analysis and Applications* 4:3(1998),245–267.

- A. Jensen and A. la Cour-Harbo. *Ripples in Mathematics: The Discrete Wavelet Transform*. Springer-Verlag, Berlin, 2001. ISBN 3-540-41662-5.

- G. Strang and T. Nguyen. *Wavelets and Filter Banks*. Wellesley–Cambridge Press, Wellesley, Massachusetts, 1996. ISBN 0-9614088-7-1.

- M. V. Wickerhauser. *Mathematics for Multimedia*. Elsevier/Academic Press, San Diego, California, 2003. ISBN 0-12-748451-5.