The Essential Norm of Operators on the Bergman Space

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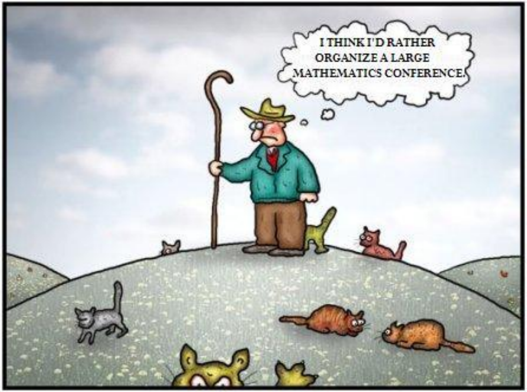
Thanks to the Organizers



The daydreams of cat herders

(Modified from the Original Dr. Fun Comic)

Thanks to Bernhard, David, Mark, Paulette and Vern for Organizing GPOTS!

This talk is based on joint work with:

Daniel Suárez

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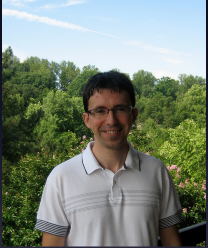

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Weighted Bergman Spaces on $\mathbb{B}_n$

- Let $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$.
- For $\alpha > -1$, we let

$$dv_\alpha(z) := c_\alpha \, (1 - |z|^2)^\alpha \, dv(z), \quad \text{with} \quad c_\alpha := \frac{\Gamma(n+\alpha+1)}{n! \, \Gamma(\alpha+1)}.$$

  The choice of c_α gives that $v_\alpha(\mathbb{B}_n) = 1$.
- For $1 < p < \infty$ the space A^p_α is the collection of holomorphic functions on $\mathbb{B}_n$ such that

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) < \infty.$$

- For $\lambda \in \mathbb{B}_n$ let $k^{(p,\alpha)}_\lambda(z) = \frac{(1-|\lambda|^2)^{\frac{n+1+\alpha}{q}}}{(1-\overline{\lambda}z)^{n+1+\alpha}}$.
- A computation shows: $\left\| k^{(p,\alpha)}_\lambda \right\|_{A^p_\alpha} \approx 1$.

Toeplitz Operators and the Toeplitz Algebra

- The projection of L^2_α onto A^2_α is given by the integral operator
$$P_\alpha(f)(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1-z\overline{w})^{n+1+\alpha}}\, dv_\alpha(w).$$
- This operator is bounded from L^p_α to A^p_α when $1 < p < \infty$ and $-1 < \alpha$.
- Let M_a denote the operator of multiplication by the function a, $M_a(f) := af$. The Toeplitz operator with symbol $a \in L^\infty$ is the operator given by
$$T_a := P_\alpha M_a.$$
- It is immediate to see that $\|T_a\|_{\mathcal{L}(A^p_\alpha)} \lesssim \|a\|_{L^\infty}$.
- More generally, for a measure μ we will define the operator
$$T_\mu f(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1-\overline{w}z)^{n+1+\alpha}}\, d\mu(w),$$
which will define an analytic function for all $f \in H^\infty$.

Toeplitz Operators and the Toeplitz Algebra

- For symbols in L^∞ we let $\mathcal{T}_{p,\alpha}$ be the C^* subalgebra of $\mathcal{L}(A^p_\alpha)$ generated by T_a.
- An important class of operators in $\mathcal{T}_{p,\alpha}$ are those that are finite sums of finite products of Toeplitz operators.
  Namely, for symbols $a_{jk} \in L^\infty$ with $1 \leq j \leq J$ and $1 \leq k \leq K$ we will need to study the operators:

$$\sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$$

- Additionally,

$$\mathcal{T}_{p,\alpha} = \overline{\left\{ \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}} : a_{jk} \in L^\infty \quad 1 \leq j \leq J \quad 1 \leq k \leq K \right\}}^{\mathcal{L}(A^p_\alpha)}$$

Geometry of the Ball

For $z \in \mathbb{B}_n$, φ_z will denote the automorpishm of $\mathbb{B}_n$ such that $\varphi_z(0) = z$. The pseudohyperbolic and hyperbolic metrics are defined by

$$\rho(z,w) := |\varphi_z(w)| \quad \text{and} \quad \beta(z,w) := \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

The hyperbolic disc centered at z of radius r is denoted by

$$D(z,r) := \{w \in \mathbb{B}_n : \beta(z,w) \leq r\} = \{w \in \mathbb{B}_n : \rho(z,w) \leq \tanh r\}.$$

Lemma (Lattices on $\mathbb{B}_n$)

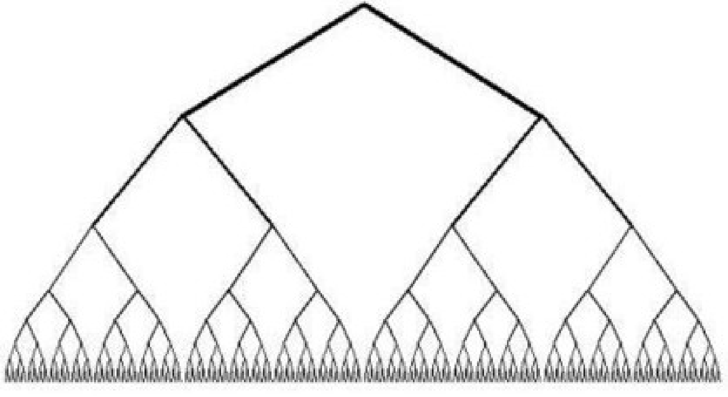
Given $r > 0$, there is a family of Borel sets $D_m \subset \mathbb{B}_n$ and points $\{w_m\}_{m=1}^{\infty}$ such that

(i) $D\left(w_m, \frac{r}{4}\right) \subset D_m \subset D\left(w_m, r\right)$ *for all* m;

(ii) $D_k \cap D_l = \emptyset$ *if* $k \neq l$;

(iii) $\bigcup_m D_m = \mathbb{B}_n$.

Geometry of the Ball



Dyadic Tree on $\mathbb{D}$

Geometry of the Ball

Note that for these sets: If $w \in D_m$ then $(1-|w|^2) \approx (1-|w_m|^2)$ and $|1-\overline{z}w| \approx |1-\overline{z}w_m|$ uniformly in $z \in \mathbb{B}_n$.

Lemma (Whitney Decompositions)

There is a positive integer $N = N(n)$ such that for any $\sigma > 0$ there is a covering of $\mathbb{B}_n$ by Borel sets $\{B_j\}$ that satisfy:

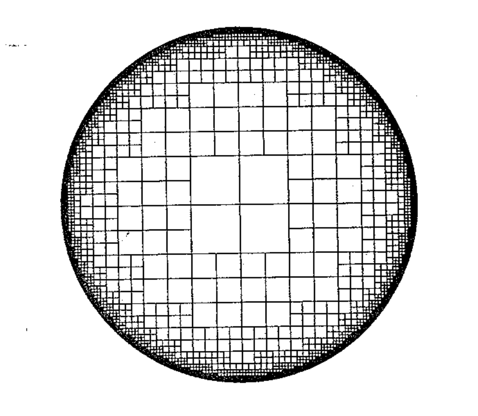
(i) $B_j \cap B_k = \emptyset$ *if* $j \neq k$;

(ii) *Every point of $\mathbb{B}_n$ is contained in at most N sets* $\Omega_\sigma(B_j) = \{z : \beta(z, B_j) \leq \sigma\}$;

(iii) *There is a constant $C(\sigma) > 0$ such that* $\operatorname{diam}_\beta B_j \leq C(\sigma)$ *for all j.*

Idea of Proof: Via the Whitney Decomposition of the unit ball $\mathbb{B}_n$, partition into dyadic "cubes." This then gives (i) immediately. The remaining points are then well known geometric facts.

Geometry of the Ball



Whitney Decomposition of $\mathbb{D}$
(Taken from *Classical and Modern Fourier Analysis* by Grafakos)

Geometry of the Ball

Let $\sigma > 0$ and k a non-negative integer. Let $\{B_j\}$ be the covering of the ball from the previous Lemma with $(k+1)\sigma$ instead of σ.
For $0 \leq i \leq k$ and $j \geq 1$ write

$$F_{0,j} = B_j \quad \text{and} \quad F_{i+1,j} = \{z : \beta(z, F_{i,j}) \leq \sigma\}.$$

Corollary

Let $\sigma > 0$ and k be a non-negative integer. For each $0 \leq i \leq k$ the family of sets $\mathcal{F}_i = \{F_{i,j} : j \geq 1\}$ forms a covering of $\mathbb{B}_n$ such that

- (i) $F_{0,j_1} \cap F_{0,j_2} = \emptyset$ *if* $j_1 \neq j_2$;
- (ii) $F_{0,j} \subset F_{1,j} \subset \cdots \subset F_{k+1,j}$ *for all* j;
- (iii) $\beta(F_{i,j}, F_{i+1,j}^c) \geq \sigma$ *for all* $0 \leq i \leq k$ *and* $j \geq 1$;
- (iv) *Every point of* $\mathbb{B}_n$ *belongs to no more than* N *elements of* $\mathcal{F}_i$;
- (v) $\operatorname{diam}_\beta F_{i,j} \leq C(k, \sigma)$ *for all* i, j.

Carleson Measures for A^p_α

A measure μ on $\mathbb{B}_n$ is a Carleson measure for A^p_α if

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \lesssim \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \quad \forall f \in A^p_\alpha.$$

Lemma (Characterizations of A^p_α Carleson Measures)

Suppose that $1 < p < \infty$ and $\alpha > -1$. Let μ be a measure on $\mathbb{B}_n$ and $r > 0$. The following quantities are equivalent, with constants that depend on n, α and r:

(1) $\|\mu\|_{\mathrm{CM}} := \sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\overline{z}w|^{2(n+1+\alpha)}} \, d\mu(w)$;

(2) $\|\imath_p\| := \inf \left\{ C : \left(\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \right)^{\frac{1}{p}} \right\}$;

(3) $\|\mu\|_{\mathrm{Geo}} := \sup_{z \in \mathbb{B}_n} \frac{\mu(D(z,r))}{\left(1-|z|^2\right)^{n+1+\alpha}}$;

(4) $\|T_\mu\|_{\mathcal{L}(A^p_\alpha)}$.

The Berezin Transform

For $S \in \mathcal{L}(A^p_\alpha)$, we define the Berezin transform by

$$B(S)(z) := \left\langle S k_z^{(p,\alpha)}, k_z^{(q,\alpha)} \right\rangle_{A^2_\alpha}.$$

- $B : \mathcal{L}(A^p_\alpha) \to L^\infty(\mathbb{B}_n)$:

$$|B(S)(z)| \leq \|S\|_{\mathcal{L}(A^p_\alpha)} \left\| k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha} \left\| k_\lambda^{(q,\alpha)} \right\|_{A^q_\alpha} \approx \|S\|_{\mathcal{L}(A^p_\alpha)}.$$

- If S is compact, then $B(S)(z) \to 0$ as $|z| \to 1$:

$$|B(S)(z)| \leq \left\| S k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha} \left\| k_\lambda^{(q,\alpha)} \right\|_{A^q_\alpha} \approx \left\| S k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha}.$$

However, $k_\lambda^{(p,\alpha)} \rightharpoonup 0$ as $|z| \to 1$ and so $\left\| S k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha} \to 0$.

The Berezin Transform

- The Berezin transform is one-to-one: Enough to show that $B(S)(z)=0 \Rightarrow S=0$.
  Set $F(z,w)=\left\langle Sk_z^{(p,\alpha)}, k_w^{(q,\alpha)}\right\rangle_{A^2_\alpha}$.
  Then $F(z,z)=0$ and F is analytic in the first variable and anti-analytic in the second variable.
  This implies that F is identically zero.
  So we have that $Sk_z^{(p,\alpha)}=0$ for all $z\in\mathbb{B}_n$, or $S=0$.
- $B(S)$ is Lipschitz conitnuous with respect to the hyperbolic metric

$$|B(S)(z_1)-B(S)(z_2)|\leq\sqrt{2}\,\|S\|_{\mathcal{L}(A^p_\alpha)}\,\beta(z_1,z_2)$$

- Range of B is not closed: $B^{-1}:B(\mathcal{L}(A^p_\alpha))\to\mathcal{L}(A^p_\alpha)$ is not bounded.

Related Results

Theorem (Axler and Zheng, Indiana Univ. Math. J. 47 (1998))

Suppose that $a_{jk} \in L^\infty(\mathbb{D})$ with $1 \leq j \leq J$ and $1 \leq k \leq K$. Let $S = \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$ The following are equivalent:

(a) *The operator S is compact on $A^2(\mathbb{D})$;*

(b) *$B(S)(z) \to 0$ as $|z| \to 1$;*

(c) *$\|Sk_z\|_{A^2_\alpha} \to 0$ as $|z| \to 1$.*

- The interesting implication is $(b) \Rightarrow (a)$;
- The same proof works in the case of the unit ball, but was done by Raimondo.

Theorem (Engliš, Ark. Mat. 30 (1992))

Let $1 < p < \infty$ and $\alpha > -1$. If S is a compact operator on A^p_α, then $S \in \mathcal{T}_{p,\alpha}$.

Main Question of Interest

From the previous Theorem and simple functional analysis we have that if S is compact on A^p_α then

$$S \in \mathcal{T}_{p,\alpha} \quad \text{and} \quad B(S)(z) \to 0 \text{ as } |z| \to 1.$$

Question (Characterizing the Compacts)

If $S \in \mathcal{T}_{p,\alpha}$ and $B(S)(z) \to 0$ as $|z| \to 1$, then is S is compact?

Yes!

- Shown to be true by Suárez for A^p when $1 < p < \infty$ and $\alpha = 0$.
- Extended to $\alpha > -1$ by Suárez, Mitkovski and BDW using some maximal ideal theory and appropriate modifications of the original proof.
- Alternate proof obtained by Mitkovski and BDW, but removing the maximal ideal theory.

Characterizations of Compactness and Essential Norm

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$ and $S \in \mathcal{L}(A^p_\alpha)$. Then S is compact if and only if $S \in \mathcal{T}_{p,\alpha}$ and $\lim_{|z|\to 1} B(S)(z) = 0$.

We can actually obtain much more precise information about the essential norm of an operator. For $S \in \mathcal{L}(A^p_\alpha)$ recall that

$$\|S\|_e = \inf\left\{\|S - Q\|_{\mathcal{L}(A^p_\alpha)} : Q \text{ is compact}\right\}.$$

We need to define other measures of the "size" of an operator $S \in \mathcal{L}(A^p_\alpha)$:

$$\begin{aligned}\mathfrak{b}_S &:= \sup_{r>0}\limsup_{|z|\to 1}\left\|M_{1_{D(z,r)}}S\right\|_{\mathcal{L}(A^p_\alpha, L^p_\alpha)}\\ \mathfrak{c}_S &:= \lim_{r\to 1}\left\|M_{1_{r\mathbb{B}_n^c}}S\right\|_{\mathcal{L}(A^p_\alpha, L^p_\alpha)}.\end{aligned}$$

In the last definition, we have that $r\mathbb{B}_n^c := \mathbb{B}_n \setminus r\mathbb{B}_n$.

Characterizations of Compactness and Essential Norm

Let $r > 0$ and let $\{w_m\}$ and D_m be the sets that form the lattice in $\mathbb{B}_n$. Define the measure

$$\mu_r = \sum_m v_\alpha(D_m)\delta_{w_m} \approx \sum_m (1 - |w_m|^2)^{\alpha+n+1}\delta_{w_m}.$$

It is well know that μ_r is a A^p_α Carleson measure, so $T_{\mu_r} : A^p_\alpha \to A^p_\alpha$ is bounded.

Lemma

$T_{\mu_r} \to Id$ on $\mathcal{L}(A^p_\alpha)$ when $r \to 0$.

Let $r > 0$ be chosen so that $\|T_{\mu_r} - Id\|_{\mathcal{L}(A^p_\alpha)} < \frac{1}{4}$, and $\mu := \mu_r$. Then set

$$\mathfrak{a}_S(\rho) := \limsup_{|z|\to 1} \sup \left\{ \|Sf\|_{A^p_\alpha} : f \in T_{\mu 1_{D(z,\rho)}}(A^p_\alpha), \|f\|_{A^p_\alpha} \leq 1 \right\}$$

and define

$$\mathfrak{a}_S := \lim_{\rho\to 1} \mathfrak{a}_S(\rho).$$

Characterizations of Compactness and Essential Norm

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$ and let $S \in \mathcal{T}_{p,\alpha}$. Then there exists constants depending only on n, p, and α such that:

$$\mathfrak{a}_S \approx \mathfrak{b}_S \approx \mathfrak{c}_S \approx \|S\|_e .$$

For the automorphism φ_z such that $\varphi_z(0) = z$ define the map

$$U_z^{(p,\alpha)} f(w) := f(\varphi_z(w)) \frac{(1 - |z|^2)^{\frac{n+1+\alpha}{p}}}{(1 - w\overline{z})^{\frac{2(n+1+\alpha)}{p}}}.$$

A standard change of variable argument and computation gives that

$$\left\| U_z^{(p,\alpha)} f \right\|_{A_\alpha^p} = \|f\|_{A_\alpha^p} \quad \forall f \in A_\alpha^p.$$

Characterizations of Compactness and Essential Norm

For $z \in \mathbb{B}_n$ and $S \in \mathcal{L}(A^p_\alpha)$ we then define the map

$$S_z := U_z^{(p,\alpha)} S (U_z^{(q,\alpha)})^*.$$

One should think of the map S_z in the following way. This is an operator on A^p_α and so it first acts as "translation" in $\mathbb{B}_n$, then the action of S, then "translation" back.

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $\alpha > -1$ and $1 < p < \infty$ and $S \in \mathcal{T}_{p,\alpha}$. Then

$$\|S\|_e \approx \sup_{\|f\|_{A^p_\alpha}=1} \limsup_{|z|\to 1} \|S_z f\|_{A^p_\alpha}.$$

Connecting the Geometry and Operator Theory

Lemma (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{T}_{p,\alpha}$, μ a Carleson measure and $\epsilon > 0$. Then there are Borel sets $F_j \subset G_j \subset \mathbb{B}_n$ such that

(i) $\mathbb{B}_n = \cup F_j$;

(ii) $F_j \cap F_k = \emptyset$ *if* $j \neq k$;

(iii) *each point of* $\mathbb{B}_n$ *lies in no more than* $N(n)$ *of the sets* G_j;

(iv) $\operatorname{diam}_\beta G_j \leq d(p, S, \epsilon)$

and

$$\left\| ST_\mu - \sum_{j=1}^{\infty} M_{1_{F_j}} ST_{1_{G_j}\mu} \right\|_{\mathcal{L}(A^p_\alpha, L^p_\alpha)} < \epsilon.$$

A Uniform Algebra and its Maximal Ideal Space

- Let $\mathcal{A}$ denote the bounded functions that are uniformly continuous from the metric space $(\mathbb{B}_n, \rho)$ into the metric space $(\mathbb{C}, |\cdot|)$.
- Associate to $\mathcal{A}$ its maximal ideal space $M_{\mathcal{A}}$ which is the set of all non-zero multiplicative linear functionals from $\mathcal{A}$ to $\mathbb{C}$.
- Since $\mathcal{A}$ is a C^* algebra we have that $\mathbb{B}_n$ is dense in $M_{\mathcal{A}}$.
- The Toeplitz operators associated to symbols in $\mathcal{A}$ are useful to study the Toeplitz algebra $\mathcal{T}_{p,\alpha}$.

Theorem (D. Suárez, M. Mitkovski and BDW)

The Toeplitz algebra $\mathcal{T}_{p,\alpha}$ is equal to the closed algebra generated by $\{T_a : a \in \mathcal{A}\}$.

A Uniform Algebra and its Maximal Ideal Space

- For an element $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$ choose a net $z_\omega \to x$.
- Form S_{z_ω} and look at the limit operator obtained when $z_\omega \to x$, denote it by S_x.

Lemma (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{L}(A^p_\alpha)$. Then $B(S)(z) \to 0$ as $|z| \to 1$ if and only if $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$.

We can extend this to compute the essential norm of an operator S in terms of S_x where $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$.

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{T}_{p,\alpha}$. Then there exists a constant $C(p, \alpha, n)$ such that

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \|S_x\|_{\mathcal{L}(A^p_\alpha)} \approx \|S\|_e .$$

The Hilbert Space Case

- More precise information can be obtained in terms of the essential norm and the essential spectral radius.
- Let $\mathcal{K}$ denote the ideal of compact operators on A^2_α.
- Recall that the Calkin algebra is given by $\mathcal{L}(A^2_\alpha)/\mathcal{K}$.
- The spectrum of S will be denoted by $\sigma(S)$, and the spectral radius will be denoted by

$$r(S) = \sup\left\{|\lambda| : \lambda \in \sigma(S)\right\}.$$

- Define the essential spectrum as the spectrum of $S + \mathcal{K}$ in the Calkin algebra, and the essential spectral radius as

$$r_e(S) = \sup\left\{|\lambda| : \lambda \in \sigma_e(S)\right\}.$$

The Hilbert Space Case

Theorem (D. Suárez, M. Mitkovski and BDW)

For $S \in \mathcal{T}_{2,\alpha}$ we have

$$\|S\|_e = \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \|S_x\|_{\mathcal{L}(A^2_\alpha)}$$

and

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} r(S_x) \leq \lim_{k \to \infty} \left(\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \left\| S_x^k \right\|_{\mathcal{L}(A^2_\alpha)}^{\frac{1}{k}} \right) = r_e(S)$$

with equality when S is essentially normal.

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $\alpha > -1$ and $S \in \mathcal{T}_{2,\alpha}$. Then

$$\|S\|_e = \sup_{\|f\|_{A^2_\alpha} = 1} \limsup_{|z| \to 1} \|S_z f\|_{A^2_\alpha}.$$

The Hilbert Space Case

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{T}_{2,\alpha}$. The following are equivalent:

(1) $\lambda \notin \sigma_e(S)$;

(2)

$$\lambda \notin \bigcup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \sigma(S_x) \quad \text{and} \quad \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \left\| (S_x - \lambda I)^{-1} \right\|_{\mathcal{L}(A^2_\alpha)} < \infty;$$

(3) *There is a number $t > 0$ depending only on λ such that*

$$\|(S_x - \lambda I) f\|_{A^2_\alpha} \geq t \|f\|_{A^2_\alpha} \quad \text{and} \quad \left\| (S_x^* - \overline{\lambda} I) f \right\|_{A^2_\alpha} \geq t \|f\|_{A^2_\alpha}$$

for all $f \in A^2_\alpha$ and $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$.

Removing the Maximal Ideal Space

Lemma (M. Mitkovski and BDW)

Let T be a finite sum of finite products of Toeplitz operators on A^2_α. For every $\epsilon > 0$ there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{F_j\}$ associated to r

$$\left\| \sum_j M_{1_{F_j}} T P_\alpha M_{1_{G_j^c}} \right\|_{\mathcal{L}(A^2_\alpha)} < \epsilon.$$

Proposition (M. Mitkovski and BDW)

Let T be a finite sum of finite products of Toeplitz operators on A^2_α. Then

$$\|T\|_e \approx \sup_{\|f\|_{A^2_\alpha}=1} \limsup_{|z|\to 1} \|T_z f\|_{A^2_\alpha}.$$

Other Directions

The Fock space $\mathcal{F}$ is the collection of holomorphic functions f on $\mathbb{C}^n$ such that

$$\|f\|_{\mathcal{F}}^2 := \int_{\mathbb{C}^n} |f(z)|^2 \, e^{-\pi|z|^2} dv(z) < \infty.$$

This is a reproducing kernel Hilbert space with $k_\lambda(z) = e^{\pi z \overline{\lambda}}$ as kernel. Similar results are true for the Fock Space.

Theorem (W. Bauer and J. Isralowitz)

Let $S \in \mathcal{L}(\mathcal{F})$. Then S is compact if and only if $S \in \mathcal{T}_2$ and $\lim_{|z|\to\infty} B(S)(z) = 0$.

Corollary (W. Bauer and J. Isralowitz)

Let $S \in \mathcal{T}_2$, then

$$\|S\|_e \approx \sup_{\|f\|_{\mathcal{F}}=1} \limsup_{|z|\to\infty} \|S_z f\|_{\mathcal{F}}.$$

Open Questions

If $S \in \mathcal{L}(A^2_\alpha)$ then $B(S) \in L^\infty$. Similarly, if $S \in \mathcal{K}(A^2_\alpha)$ then $B(S) \to 0$ as $|z| \to 1$. And, even better, $S \in \mathcal{K}(A^2_\alpha)$ if and only if $S \in \mathcal{T}_{2,\alpha}$ and $B(S) \to 0$ as $|z| \to 1$.

Question

Can we characterize the Schatten class operators on A^2_α as those that belong to the Toeplitz algebra $\mathcal{T}_{2,\alpha}$ and an integrability condition on the Berezin transform $B(S)(z)$?

One can show that if $S \in \mathcal{S}_p$ then

$$\|B(S)\|_{L^p(\mathbb{B}_n;\lambda_n)} := \left(\int_{\mathbb{B}_n} |B(S)(z)|^p \, d\lambda_n(z)\right)^{\frac{1}{p}} \lesssim \|S\|_{\mathcal{S}_p}.$$

Proposition (M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$. If $S \in \mathcal{S}_p$, then $B(S) \in L^p(\mathbb{B}_n;\lambda_n)$ and $S \in \mathcal{T}_{2,\alpha}$.

Open Questions

Let Ω be a bounded symmetric domain in $\mathbb{C}^n$. These are Hermitian symmetric spaces with a complete Riemannian metric given by the Bergman metric. For each $a \in \Omega$ there is a biholomorphic automorphism φ_a that interchanges 0 and a.

Let $A^2(\Omega)$ denote the Bergman space of analytic functions on Ω that are square integrable with respect to volume measure. This space has a reproducing kernel K_a, and relates to the automorphisms by

$$K_w(z) = K_{\varphi(w)}(\varphi(z)) J_c\varphi(w) \overline{J_c\varphi(w)}$$

Conjecture (M. Mitkovski and BDW)

Let $S \in \mathcal{T}_2$, then

$$\|S\|_e \approx \sup_{\|f\|_{A^2(\Omega)}=1} \limsup_{z \to \partial\Omega} \|S_z f\|_{A^2(\Omega)}.$$

Thank You!