Corona Theorems for Multiplier Algebras on $\mathbb{B}_n$

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Talk Outline

- Motivation of the Problem
    - Review of the Corona Problem in One Variable
    - Besov-Sobolev Spaces and Multiplier Algebras
    - Baby Corona versus Corona
    - Toeplitz Corona Theorem
- Main Result and Sketch of Proof
- Further Results and Future Directions

The Corona Problem for $H^\infty(\mathbb{D})$

- The Banach algebra $H^\infty(\mathbb{D})$ is the collection of all analytic functions on the disc such that

$$\|f\|_{H^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}} |f(z)| < \infty$$

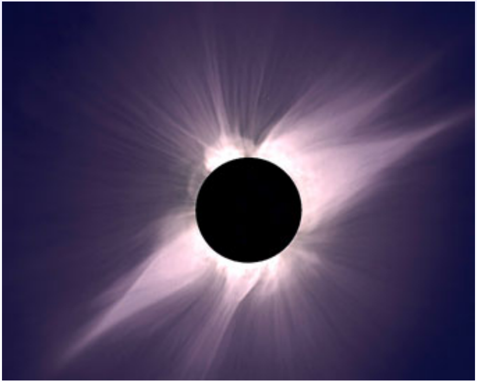
- To each $z \in \mathbb{D}$ we can associate a multiplicative linear functional on $H^\infty(\mathbb{D})$ (point evaluation at z)

$$\varphi_z(f) := f(z)$$

- Let $\triangle$ denote the maximal ideal space of $H^\infty(\mathbb{D})$.
(Maximal ideals = kernels of multiplicative linear functionals)

The Corona Problem for $H^\infty(\mathbb{D})$

In 1941, Kakutani asked if there was a Corona in the maximal ideal space $\triangle$ of $H^\infty(\mathbb{D})$, i.e. whether or not the disc $\mathbb{D}$ was dense in $\triangle$. One then defines the Corona of the algebra to be $\triangle \setminus \overline{\mathbb{D}}$.



The Corona Problem for $H^{\infty}(\mathbb{D})$

In 1962, Lennart Carleson demonstrated the absence of a Corona by showing that if $\{g_j\}_{j=1}^{N}$ is a finite set of functions in $H^{\infty}(\mathbb{D})$ satisfying

$$0 < \delta \leq \sum_{j=1}^{N} |g_j(z)|^2 \leq 1, \qquad z \in \mathbb{D},$$

then there are functions $\{f_j\}_{j=1}^{N}$ in $H^{\infty}(\mathbb{D})$ with

$$\sum_{j=1}^{N} f_j(z) g_j(z) = 1, \qquad z \in \mathbb{D}.$$

Moreover,

$$\sum_{j=1}^{N} \|f_j\|_{H^{\infty}(\mathbb{D})} \leq C(\delta).$$

Extensions of the Corona Problem

The point of departure for many generalizations of Carleson's Corona Theorem is the following:

Observation

$H^\infty(\mathbb{D})$ is the (pointwise) multiplier algebra of the classical Hardy space $H^2(\mathbb{D})$ on the unit disc.

Namely, let $M_{H^2}(\mathbb{D})$ denote the class of functions φ such that

$$\|\varphi f\|_{H^2(\mathbb{D})} \leq C\|f\|_{H^2(\mathbb{D})}, \quad \forall f \in H^2(\mathbb{D}). \qquad (\dagger)$$

with $\|\varphi\|_{M_{H^2}(\mathbb{D})} = \inf\{C : (\dagger) \text{ holds}\}$. Then $\varphi \in H^\infty(\mathbb{D})$ if and only if $\varphi \in M_{H^2}(\mathbb{D})$ and,

$$\|\varphi\|_{M_{H^2}(\mathbb{D})} = \|\varphi\|_{H^\infty(\mathbb{D})}.$$

Besov-Sobolev Spaces

- The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions f on the unit ball $\mathbb{B}_n$ such that

$$\left\{\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|^2+\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma}f^{(m)}(z)\right|^2 d\lambda_n(z)\right\}^{\frac{1}{2}}<\infty,$$

  where $d\lambda_n(z)=\left(1-|z|^2\right)^{-n-1}dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m+\sigma>\frac{n}{2}$.
- Various choices of σ give important examples of classical function spaces:
  - $\sigma=0$: Corresponds to the Dirichlet Space;
  - $\sigma=\frac{1}{2}$: Drury-Arveson Hardy Space;
  - $\sigma=\frac{n}{2}$: Classical Hardy Space;
  - $\sigma>\frac{n}{2}$: Bergman Spaces.

Besov-Sobolev Spaces

- The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces.
- Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{B_2^\sigma}$$

- It isn't too difficult to compute (or show) that the kernel function k_λ is given by

$$k_\lambda(z) = \frac{1}{\left(1 - \overline{\lambda} z\right)^{2\sigma}}$$

  - $\sigma = \frac{1}{2}$: Drury-Arveson Hardy Space; $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$
  - $\sigma = \frac{n}{2}$: Classical Hardy Space; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^n}$
  - $\sigma = \frac{n+1}{2}$: Bergman Space; $k_\lambda(z) = \frac{1}{(1-\overline{\lambda}z)^{n+1}}$

Multiplier Algebras of Besov-Sobolev Spaces $M_{B_2^\sigma}(\mathbb{B}_n)$

- We are interested in the multiplier algebras, $M_{B_2^\sigma}(\mathbb{B}_n)$, for $B_2^\sigma(\mathbb{B}_n)$. A function φ belongs to $M_{B_2^\sigma}(\mathbb{B}_n)$ if

$$\|\varphi f\|_{B_2^\sigma(\mathbb{B}_n)} \leq C\|f\|_{B_2^\sigma(\mathbb{B}_n)} \quad \forall f \in B_2^\sigma(\mathbb{B}_n)$$
$$\|\varphi\|_{M_{B_2^\sigma}(\mathbb{B}_n)} = \inf\{C : \text{above inequality holds}\}.$$

- It is easy to see that $M_{B_2^\sigma}(\mathbb{B}_n) = H^\infty(\mathbb{B}_n) \cap \mathcal{X}_2^\sigma(\mathbb{B}_n)$.
Where $\mathcal{X}_2^\sigma(\mathbb{B}_n)$ is the collection of functions φ such that for all $f \in B_2^\sigma(\mathbb{B}_n)$:

$$\int_{\mathbb{B}_n} |f(z)|^2 \left|\left(1-|z|^2\right)^{m+\sigma} \varphi^{(m)}(z)\right|^2 d\lambda_n(z) \leq C\|f\|^2_{B_2^\sigma(\mathbb{B}_n)}, \qquad (\ddagger)$$

with $\|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)} = \inf\{C : (\ddagger) \text{ holds}\}$.
Thus, we have

$$\|\varphi\|_{M_{B_2^\sigma}(\mathbb{B}_n)} \approx \|\varphi\|_{H^\infty(\mathbb{B}_n)} + \|\varphi\|_{\mathcal{X}_2^\sigma(\mathbb{B}_n)}.$$

The Corona Problem for $M_{B_2^\sigma}(\mathbb{B}_n)$

We wish to study a generalization of Carleson's Corona Theorem to higher dimensions and additional function spaces.

Question (Corona Problem)

Given $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying* $0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1$ *for all* $z \in \mathbb{B}_n$. *Does there exist a constant* $C_{n,\sigma,N,\delta}$ *and functions* $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^N \|f_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \\ \sum_{j=1}^N g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n? \end{aligned}$$

The Baby Corona Problem

It is easy to see that the Corona Problem for $M_{B_2^\sigma}(\mathbb{B}_n)$ implies a "simpler" question that one can consider.

Question (Baby Corona Problem)

Given $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying* $0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1$ *for all* $z \in \mathbb{B}_n$ *and* $h \in B_2^\sigma(\mathbb{B}_n)$. *Does there exist a constant* $C_{n,\sigma,N,\delta}$ *and functions* $f_1, \ldots, f_N \in B_2^\sigma(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^N \|f_j\|_{B_2^\sigma(\mathbb{B}_n)}^2 &\leq C_{n,\sigma,N,\delta} \|h\|_{B_2^\sigma(\mathbb{B}_n)}^2, \\ \sum_{j=1}^N g_j(z) f_j(z) &= h(z), \qquad z \in \mathbb{B}_n? \end{aligned}$$

Baby Corona Theorem for $B_p^\sigma(\mathbb{B}_n)$

Theorem (Ş. Costea, E. Sawyer, BDW (2008))

Let $0 \leq \sigma$ *and* $1 < p < \infty$. *Given* $g_1, \ldots, g_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |g_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

there is a constant $C_{n,\sigma,N,p,\delta}$ *such that for each* $h \in B_p^\sigma(\mathbb{B}_n)$ *there are* $f_1, \ldots, f_N \in B_p^\sigma(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^{N} \|f_j\|_{B_p^\sigma(\mathbb{B}_n)}^p &\leq C_{n,\sigma,N,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}^p, \\ \sum_{j=1}^{N} g_j(z) f_j(z) &= h(z), \qquad z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

Corollary (Ş. Costea, E. Sawyer, BDW (2008))

Let $0 \leq \sigma \leq \frac{1}{2}$. *Given* $g_1, \ldots, g_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |g_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

there is a constant $C_{n,\sigma,N,\delta}$ *and there are functions* $f_1, \ldots, f_N \in M_{B_2^\sigma}(\mathbb{B}_n)$ *satisfying*

$$\begin{aligned} \sum_{j=1}^{N} \|f_j\|_{M_{B_2^\sigma}(\mathbb{B}_n)} &\leq C_{n,\sigma,N,\delta} \\ \sum_{j=1}^{N} g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n. \end{aligned}$$

The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

The proof of this Corollary follows from the main Theorem very easily.

- When $0 \leq \sigma \leq \frac{1}{2}$ the spaces $B_2^\sigma(\mathbb{B}_n)$ are reproducing kernel Hilbert spaces with a complete Nevanlinna-Pick kernel.
- By the Toeplitz Corona Theorem, we then have that the Baby Corona Problem is equivalent to the full Corona Problem. The result then follows.

An additional corollary of the above result is the following:

Corollary

For $0 \leq \sigma \leq \frac{1}{2}$, the unit ball $\mathbb{B}_n$ is dense in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$.

This is because the density of the the unit ball $\mathbb{B}_n$ in the maximal ideal space of $M_{B_2^\sigma}(\mathbb{B}_n)$ is equivalent to the Corona Theorem above.

Sketch of Proof of the Baby Corona Theorem

Given $g_1, \ldots, g_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ satisfying

$$0 < \delta \leq \sum_{j=1}^{N} |g_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

- Set $\varphi_j(z) = \frac{\overline{g_j(z)}}{\sum_j |g_j(z)|^2} h(z)$. We have that $\sum_{j=1}^N g_j(z)\varphi_j(z) = h(z)$.
  - This solution is smooth and satisfies the correct estimates, but is far from analytic.
- In order to have an analytic solution, we will need to solve a sequence of $\overline{\partial}$-equations:
  - For η a $\overline{\partial}$-closed $(0, q)$ form, we want to solve the equation

    $$\overline{\partial}\psi = \eta$$

    for ψ a $(0, q-1)$ form.
  - To accomplish this, we will use the Koszul complex. This gives an algorithmic way of solving the $\overline{\partial}$-equations for each $(0, q)$ with $1 \leq q \leq n$ after starting with a $(0, n)$ form.

Sketch of Proof of the Baby Corona Theorem

- This produces a correction to the initial guess of φ_j, call it ξ_j, and set $f_j = \varphi_j - \xi_j$. By the Koszul complex we will have that each f_j is in fact analytic.
  Algebraic properties of the Koszul complex give that $\sum_j f_j g_j = h$. However, now the estimates that we seek are in doubt.

- To guarantee the estimates, we have to look closer at the solution operator to the $\overline{\partial}$-equation on $\overline{\partial}$-closed $(0, q)$ forms. Following the work of Øvrelid and Charpentier, one can compute that the solution operator is an integral operator that that takes $(0, q)$ forms to $(0, q-1)$ forms with integral kernel:

$$\frac{(1 - w\overline{z})^{n-q}\left(1 - |w|^2\right)^{q-1}}{\triangle(w, z)^n}(\overline{w}_j - \overline{z}_j) \quad \forall\, 1 \leq q \leq n.$$

Here $\triangle(w, z) = |1 - w\overline{z}|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right)$.

Sketch of Proof of the Baby Corona Theorem

- One then needs to show that these solution operators map the Besov-Sobolev spaces $B_\sigma^p(\mathbb{B}_n)$ to themselves. This is accomplished by a couple of key facts:
    - The Besov-Sobolev spaces are very "flexible" in terms of the norm that one can use. One need only take the parameter m sufficiently high.
    - We show that these operators are very well behaved on "real variable" versions of the space $B_\sigma^p(\mathbb{B}_n)$. These, of course, contain the space that we are interested in.
    - To show that the solution operators are bounded on $L^p(\mathbb{B}_n; dV)$ the original proof uses the Schur Test. To handle the boundedness on $B_\sigma^p(\mathbb{B}_n)$, we can also use the Schur test but this requires more work to handle the derivative. This is key to the proof.
- Certain properties of the unit ball (i.e., symmetries) are exploited to make some of these computations easier.

The $H^\infty(\mathbb{B}_n)$ Corona Problem

When $\sigma = \frac{n}{2}$ (the classical Hardy space $H^2(\mathbb{B}_n)$), we have a weaker version of the Corona Problem that we can prove.

Theorem (Ş. Costea, E. Sawyer, BDW (2008))

Given $g_1, \ldots, g_N \in H^\infty(\mathbb{B}_n)$ satisfying $0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1$ for all $z \in \mathbb{B}_n$. Then there is a constant $C_{n,N,\delta}$ and there are functions $f_1, \ldots, f_N \in BMOA(\mathbb{B}_n)$ satisfying

$$\begin{aligned} \sum_{j=1}^N \|f_j\|_{BMOA(\mathbb{B}_n)} &\leq C_{n,N,\delta} \\ \sum_{j=1}^N g_j(z) f_j(z) &= 1, \qquad z \in \mathbb{B}_n. \end{aligned}$$

This gives another proof of a famous theorem of Varopoulos.

Baby Corona for $H^\infty(\mathbb{B}_n)$ versus Corona for $H^\infty(\mathbb{B}_n)$

We know that the Corona Problem always implies the Baby Corona Problem. By the Toeplitz Corona Theorem, we know that, under certain conditions on the reproducing kernel, these problems are in fact equivalent. But, what happens if we don't have these conditions?

Theorem (Equivalence between Corona and Baby Corona, (Amar 2003))

Let $\{g_j\}_{j=1}^N \subseteq H^\infty(\mathbb{B}_n)$. Then there exists $\{f_j\}_{j=1}^N \subseteq H^\infty(\mathbb{B}_n)$ with

$$\sum_{j=1}^N f_j(z)g_j(z) = 1 \quad \forall z \in \mathbb{B}_n \quad \text{and} \quad \sum_{j=1}^N \|g_j\|_{H^\infty(\mathbb{B}_n)} \leq \frac{1}{\delta}$$

if and only if

$$\mathcal{M}_g^\mu (\mathcal{M}_g^\mu)^* \geq \delta^2 I_\mu$$

for all probability measures μ on $\partial\mathbb{B}_n$.

Baby Corona for $H^\infty(\mathbb{B}_n)$ versus Corona for $H^\infty(\mathbb{B}_n)$

This is a great theorem since it suggests how to attack the Corona Problem for $H^\infty(\mathbb{B}_n)$. But, the difficulty is that one must solve the Baby Corona Problem for **every** probability measure on $\partial\mathbb{B}_n$.
Instead, it is possible to reduce this to a class of probability measures for which the methods of harmonic analysis and operator theory are more amenable.

Theorem (Trent, BDW (2008))

Assume that $\mathcal{M}_g^H \mathcal{M}_g^{H*} \geq \delta^2 I_H$ for all $H \in \mathcal{H}$. Then there exists a $f_1, \ldots, f_N \in H^\infty(\mathbb{B}_n)$, so that

$$\sum_{j=1}^N f_j(z) g_j(z) = 1 \; \forall z \in \mathbb{B}_n \quad \text{and} \quad \sum_{j=1}^N \|f_j\|_{H^\infty(\mathbb{B}_n)} \leq \frac{1}{\delta}.$$

This reduces the $H^\infty(\mathbb{B}_n)$ Corona Problem to a certain "weighted" Baby Corona Problem.

Open Problems and Future Directions

1. Does the algebra $H^{\infty}(\mathbb{B}_n)$ of bounded analytic functions on the ball have a Corona in its maximal ideal space?
2. Does the Corona Theorem for the multiplier algebra of the Drury-Arveson space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ extend to more general domains in $\mathbb{C}^n$?
3. Can we prove a Corona Theorem for *any* algebra in higher dimensions that is not the multiplier algebra of a Hilbert space with the complete Nevanlinna-Pick property? Any $\frac{1}{2} < \sigma \leq \frac{n}{2}$ would be extremely interesting.
4. Can one prove the equivalence between a "weakened" version of the Baby Corona Problem and the Corona Problem when $\frac{1}{2} < \sigma < \frac{n}{2}$? This would be useful to approach the above problem.

Thank You!