The Essential Norm of Operators on the Bergman Space

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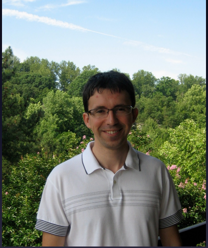
This talk is based on joint work with:



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Weighted Bergman Spaces on $\mathbb{B}_n$

- Let $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$.
- For $\alpha > -1$, we let

$$dv_\alpha(z) := c_\alpha \, (1-|z|^2)^\alpha \, dv(z), \quad \text{with} \quad c_\alpha := \frac{\Gamma(n+\alpha+1)}{n! \, \Gamma(\alpha+1)}.$$

  The choice of c_α gives that $v_\alpha(\mathbb{B}_n) = 1$.
- For $1 < p < \infty$ the space A^p_α is the collection of holomorphic functions on $\mathbb{B}_n$ such that

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) < \infty.$$

- For $\lambda \in \mathbb{B}_n$ let $k_\lambda^{(p,\alpha)}(z) = \frac{(1-|\lambda|^2)^{\frac{n+1+\alpha}{q}}}{(1-\overline{\lambda}z)^{n+1+\alpha}}$.
- A computation shows: $\left\| k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha} \approx 1$.

Toeplitz Operators and the Toeplitz Algebra

- The projection of L^2_α onto A^2_α is given by the integral operator
$$P_\alpha(f)(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1 - z\overline{w})^{n+1+\alpha}} dv_\alpha(w).$$
- This operator is bounded from L^p_α to A^p_α when $1 < p < \infty$ and $-1 < \alpha$.
- Let M_a denote the operator of multiplication by the function a, $M_a(f) := af$. The Toeplitz operator with symbol $a \in L^\infty$ is the operator given by
$$T_a := P_\alpha M_a.$$
- It is immediate to see that $\|T_a\|_{\mathcal{L}(A^p_\alpha)} \lesssim \|a\|_{L^\infty}$.
- More generally, for a measure μ we will define the operator
$$T_\mu f(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \overline{w}z)^{n+1+\alpha}} d\mu(w),$$
which will define an analytic function for all $f \in H^\infty$.

Toeplitz Operators and the Toeplitz Algebra

- For symbols in L^∞ we let $\mathcal{T}_{p,\alpha}$ be the C^* subalgebra of $\mathcal{L}(A^p_\alpha)$ generated by T_a.
- An important class of operators in $\mathcal{T}_{p,\alpha}$ are those that are finite sums of finite products of Toeplitz operators.
  Namely, for symbols $a_{jk} \in L^\infty$ with $1 \leq j \leq J$ and $1 \leq k \leq K$ we will need to study the operators:

$$\sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$$

- Additionally,

$$\mathcal{T}_{p,\alpha} = \overline{\left\{ \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}} : a_{jk} \in L^\infty \quad 1 \leq j \leq J \quad 1 \leq k \leq K \right\}}^{\mathcal{L}(A^p_\alpha)}$$

Geometry of the Ball

For $z \in \mathbb{B}_n$, φ_z will denote the automorpishm of $\mathbb{B}_n$ such that $\varphi_z(0) = z$. The pseudohyperbolic and hyperbolic metrics are defined by

$$\rho(z,w) := |\varphi_z(w)| \quad \text{and} \quad \beta(z,w) := \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

The hyperbolic disc centered at z of radius r is denoted by

$$D(z,r) := \{w \in \mathbb{B}_n : \beta(z,w) \leq r\} = \{w \in \mathbb{B}_n : \rho(z,w) \leq \tanh r\}.$$

Lemma (Lattices on $\mathbb{B}_n$)

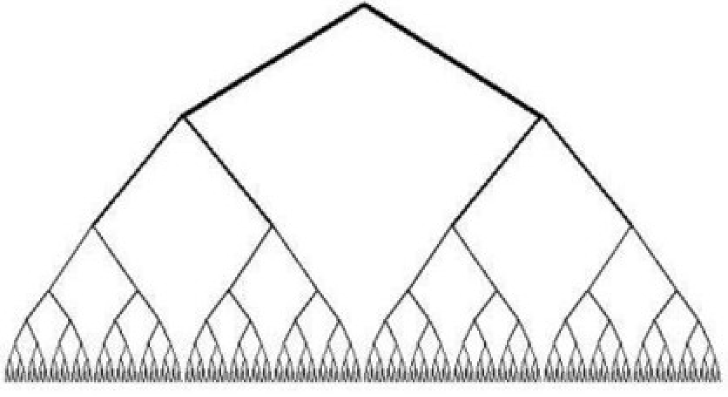
Given $r > 0$, there is a family of Borel sets $D_m \subset \mathbb{B}_n$ and points $\{w_m\}_{m=1}^{\infty}$ such that

(i) $D\left(w_m, \frac{r}{4}\right) \subset D_m \subset D\left(w_m, r\right)$ *for all* m;

(ii) $D_k \cap D_l = \emptyset$ *if* $k \neq l$;

(iii) $\bigcup_m D_m = \mathbb{B}_n$.

Geometry of the Ball



Dyadic Tree on $\mathbb{D}$

Geometry of the Ball

Note that for these sets: If $w \in D_m$ then $(1-|w|^2) \approx (1-|w_m|^2)$ and $|1-\overline{z}w| \approx |1-\overline{z}w_m|$ uniformly in $z \in \mathbb{B}_n$.

Lemma (Whitney Decompositions)

There is a positive integer $N = N(n)$ such that for any $\sigma > 0$ there is a covering of $\mathbb{B}_n$ by Borel sets $\{B_j\}$ that satisfy:

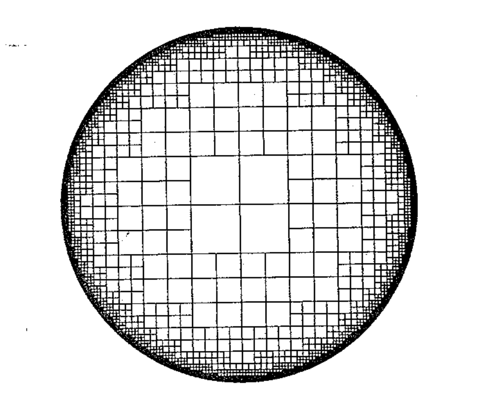
(i) $B_j \cap B_k = \emptyset$ *if* $j \neq k$;

(ii) *Every point of* $\mathbb{B}_n$ *is contained in at most* N *sets* $\Omega_\sigma(B_j) = \{z : \beta(z, B_j) \leq \sigma\}$;

(iii) *There is a constant* $C(\sigma) > 0$ *such that* $\operatorname{diam}_\beta B_j \leq C(\sigma)$ *for all* j.

Idea of Proof: Via the Whitney Decomposition of the unit ball $\mathbb{B}_n$, partition into cubes or hyperbolic balls. This then gives (i) immediately. The remaining points are then well known geometric facts.

Geometry of the Ball



Whitney Decomposition of $\mathbb{D}$
(Taken from *Classical and Modern Fourier Analysis* by Grafakos)

Geometry of the Ball

Let $\sigma > 0$ and k a non-negative integer. Let $\{B_j\}$ be the covering of the ball from the previous Lemma with $(k+1)\sigma$ instead of σ.
For $0 \leq i \leq k$ and $j \geq 1$ write

$$F_{0,j} = B_j \quad \text{and} \quad F_{i+1,j} = \{z : \beta(z, F_{i,j}) \leq \sigma\}.$$

Corollary

Let $\sigma > 0$ and k be a non-negative integer. For each $0 \leq i \leq k$ the family of sets $\mathcal{F}_i = \{F_{i,j} : j \geq 1\}$ forms a covering of $\mathbb{B}_n$ such that

(i) $F_{0,j_1} \cap F_{0,j_2} = \emptyset$ *if* $j_1 \neq j_2$;

(ii) $F_{0,j} \subset F_{1,j} \subset \cdots \subset F_{k+1,j}$ *for all* j;

(iii) $\beta(F_{i,j}, F_{i+1,j}^c) \geq \sigma$ *for all* $0 \leq i \leq k$ *and* $j \geq 1$;

(iv) *Every point of* $\mathbb{B}_n$ *belongs to no more than* N *elements of* $\mathcal{F}_i$;

(v) $\operatorname{diam}_\beta F_{i,j} \leq C(k, \sigma)$ *for all* i, j.

Carleson Measures for A^p_α

A measure μ on $\mathbb{B}_n$ is a Carleson measure for A^p_α if

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \lesssim \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \quad \forall f \in A^p_\alpha.$$

Lemma (Characterizations of A^p_α Carleson Measures)

Suppose that $1 < p < \infty$ and $\alpha > -1$. Let μ be a measure on $\mathbb{B}_n$ and $r > 0$. The following quantities are equivalent, with constants that depend on n, α and r:

(1) $\|\mu\|_{\mathrm{CM}} := \sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\overline{z}w|^{2(n+1+\alpha)}} \, d\mu(w)$;

(2) $\|\imath_p\| := \inf\left\{ C : \left(\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z)\right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z)\right)^{\frac{1}{p}} \right\}$;

(3) $\|\mu\|_{\mathrm{Geo}} := \sup_{z\in\mathbb{B}_n} \frac{\mu(D(z,r))}{\left(1-|z|^2\right)^{n+1+\alpha}}$;

(4) $\|T_\mu\|_{\mathcal{L}(A^p_\alpha)}$.

The Berezin Transform

For $S \in \mathcal{L}(A^p_\alpha)$, we define the Berezin transform by

$$B(S)(z) := \left\langle S k_z^{(p,\alpha)}, k_z^{(q,\alpha)} \right\rangle_{A^2_\alpha}.$$

- $B : \mathcal{L}(A^p_\alpha) \to L^\infty(\mathbb{B}_n)$:

$$|B(S)(z)| \leq \|S\|_{\mathcal{L}(A^p_\alpha)} \left\| k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha} \left\| k_\lambda^{(q,\alpha)} \right\|_{A^q_\alpha} \approx \|S\|_{\mathcal{L}(A^p_\alpha)}.$$

- If S is compact, then $B(S)(z) \to 0$ as $|z| \to 1$:

$$|B(S)(z)| \leq \left\| S k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha} \left\| k_\lambda^{(q,\alpha)} \right\|_{A^q_\alpha} \approx \left\| S k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha}.$$

However, $k_\lambda^{(p,\alpha)} \rightharpoonup 0$ as $|z| \to 1$ and so $\left\| S k_\lambda^{(p,\alpha)} \right\|_{A^p_\alpha} \to 0$.

The Berezin Transform

- The Berezin transform is one-to-one: Enough to show that $B(S)(z) = 0 \Rightarrow S = 0$.

  Set $F(z, w) = \left\langle S k_z^{(p,\alpha)}, k_w^{(q,\alpha)} \right\rangle_{A^2_\alpha}$.

  Then $F(z, z) = 0$ and F is analytic in the second variable and anti-analytic in the first variable.

  This implies that F is identically zero.

  So we have that $S k_z^{(p,\alpha)} = 0$ for all $z \in \mathbb{B}_n$, or $S = 0$.
- $B(S)$ is Lipschitz conitnuous with respect to the hyperbolic metric

$$|B(S)(z_1) - B(S)(z_2)| \leq \sqrt{2} \, \|S\|_{\mathcal{L}(A^p_\alpha)} \, \beta(z_1, z_2)$$

- Range of B is not closed: $B^{-1} : B(\mathcal{L}(A^p_\alpha)) \to \mathcal{L}(A^p_\alpha)$ is not bounded.

Related Results

Theorem (Axler and Zheng, Indiana Univ. Math. J. 47 (1998))

Suppose that $a_{jk} \in L^\infty(\mathbb{D})$ *with* $1 \leq j \leq J$ *and* $1 \leq k \leq K$. *Let* $S = \sum_{j=1}^{J} \prod_{k=1}^{K} T_{a_{jk}}$ *The following are equivalent:*

(a) *The operator* S *is compact on* $A^2(\mathbb{D})$;

(b) $B(S)(z) \to 0$ *as* $|z| \to 1$;

(c) $\|Sk_z\|_{A^2_\alpha} \to 0$ *as* $|z| \to 1$.

- The interesting implication is $(b) \Rightarrow (a)$;
- The same proof works in the case of the unit ball, but was done by Raimondo.

Theorem (Engliš, Ark. Mat. 30 (1992))

Let $1 < p < \infty$ *and* $\alpha > -1$. *If* S *is a compact operator on* A^p_α, *then* $S \in \mathcal{T}_{p,\alpha}$.

Main Question of Interest

From the previous Theorem and simple functional analysis we have that if S is compact on A^p_α then

$$S \in \mathcal{T}_{p,\alpha} \quad \text{and} \quad B(S)(z) \to 0 \text{ as } |z| \to 1.$$

Question (Characterizing the Compacts)

If $S \in \mathcal{T}_{p,\alpha}$ and $B(S)(z) \to 0$ as $|z| \to 1$, then is S is compact?

Yes!

- Shown to be true by Suárez for A^p when $1 < p < \infty$ and $\alpha = 0$.
- Extended to $\alpha > -1$ by Suárez, Mitkovski and BDW using some maximal ideal theory and appropriate modifications of the original proof.
- Alternate proof obtained by Mitkovski and BDW, but removing the maximal ideal theory.

Characterizations of Compactness and Essential Norm

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$ and $S \in \mathcal{L}(A^p_\alpha)$. Then S is compact if and only if $S \in \mathcal{T}_{p,\alpha}$ and $\lim_{|z|\to 1} B(S)(z) = 0$.

We can actually obtain much more precise information about the essential norm of an operator. For $S \in \mathcal{L}(A^p_\alpha)$ recall that

$$\|S\|_e = \inf\left\{\|S - Q\|_{\mathcal{L}(A^p_\alpha)} : Q \text{ is compact}\right\}.$$

We need to define other measures of the "size" of an operator $S \in \mathcal{L}(A^p_\alpha)$:

$$\begin{aligned}
\mathfrak{b}_S &:= \sup_{r>0} \limsup_{|z|\to 1} \left\| M_{1_{D(z,r)}} S \right\|_{\mathcal{L}(A^p_\alpha, L^p_\alpha)} \\
\mathfrak{c}_S &:= \lim_{r\to 1} \left\| M_{1_{\mathbb{B}_n \setminus r\mathbb{B}_n}} S \right\|_{\mathcal{L}(A^p_\alpha, L^p_\alpha)}.
\end{aligned}$$

Characterizations of Compactness and Essential Norm

Let $r > 0$ and let $\{w_m\}$ and D_m be the sets that form the lattice in $\mathbb{B}_n$. Define the measure

$$\mu_r = \sum_m v_\alpha(D_m)\delta_{w_m} \approx \sum_m (1 - |w_m|^2)^{\alpha+n+1}\delta_{w_m}.$$

It is well know that μ_r is a A^p_α Carleson measure, so $T_{\mu_r} : A^p_\alpha \to A^p_\alpha$ is bounded.

Lemma

$T_{\mu_r} \to Id$ on $\mathcal{L}(A^p_\alpha)$ when $r \to 0$.

Let $r > 0$ be chosen so that $\| T_{\mu_r} - Id\|_{\mathcal{L}(A^p_\alpha)} < \frac{1}{4}$, and $\mu := \mu_r$. Then set

$$\mathfrak{a}_S(\rho) := \limsup_{|z|\to 1} \sup \left\{ \|Sf\|_{A^p_\alpha} : f \in T_{\mu 1_{D(z,\rho)}}(A^p_\alpha), \|f\|_{A^p_\alpha} \leq 1 \right\}$$

and define

$$\mathfrak{a}_S := \lim_{\rho\to 1} \mathfrak{a}_S(\rho).$$

Characterizations of Compactness and Essential Norm

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$ and let $S \in \mathcal{T}_{p,\alpha}$. Then there exists constants depending only on n, p, and α such that:

$$\mathfrak{a}_S \approx \mathfrak{b}_S \approx \mathfrak{c}_S \approx \|S\|_e .$$

For the automorphism φ_z such that $\varphi_z(0) = z$ define the map

$$U_z^{(p,\alpha)} f(w) := f(\varphi_z(w)) \frac{\left(1 - |z|^2\right)^{\frac{n+1+\alpha}{p}}}{(1 - w\overline{z})^{\frac{2(n+1+\alpha)}{p}}}.$$

A standard change of variable argument and computation gives that

$$\left\| U_z^{(p,\alpha)} f \right\|_{A_\alpha^p} = \|f\|_{A_\alpha^p} \quad \forall f \in A_\alpha^p.$$

Characterizations of Compactness and Essential Norm

For $z \in \mathbb{B}_n$ and $S \in \mathcal{L}(A^p_\alpha)$ we then define the map

$$S_z := U_z^{(p,\alpha)} S (U_z^{(q,\alpha)})^*.$$

One should think of the map S_z in the following way. This is an operator on A^p_α and so it first acts as "translation" in $\mathbb{B}_n$, then the action of S, then "translation" back.

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $\alpha > -1$ and $1 < p < \infty$ and $S \in \mathcal{T}_{p,\alpha}$. Then

$$\|S\|_e \approx \sup_{\|f\|_{A^p_\alpha}=1} \limsup_{|z|\to 1} \|S_z f\|_{A^p_\alpha}.$$

Connecting the Geometry and Operator Theory

Lemma (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{T}_{p,\alpha}$, μ a Carleson measure and $\epsilon > 0$. Then there are Borel sets $F_j \subset G_j \subset \mathbb{B}_n$ such that

(i) $\mathbb{B}_n = \cup F_j$;

(ii) $F_j \cap F_k = \emptyset$ *if* $j \neq k$;

(iii) *each point of* $\mathbb{B}_n$ *lies in no more than* $N(n)$ *of the sets* G_j;

(iv) $\operatorname{diam}_\beta G_j \leq d(p, S, \epsilon)$

and

$$\left\| ST_\mu - \sum_{j=1}^{\infty} M_{1_{F_j}} ST_{1_{G_j}\mu} \right\|_{\mathcal{L}(A^p_\alpha, L^p_\alpha)} < \epsilon.$$

A Uniform Algebra and its Maximal Ideal Space

- Let $\mathcal{A}$ denote the bounded functions that are uniformly continuous from the metric space $(\mathbb{B}_n, \rho)$ into the metric space $(\mathbb{C}, |\cdot|)$.
- Associate to $\mathcal{A}$ its maximal ideal space $M_{\mathcal{A}}$ which is the set of all non-zero multiplicative linear functionals from $\mathcal{A}$ to $\mathbb{C}$.
- Since $\mathcal{A}$ is a C^* algebra we have that $\mathbb{B}_n$ is dense in $M_{\mathcal{A}}$.
- The Toeplitz operators associated to symbols in $\mathcal{A}$ are useful to study the Toeplitz algebra $\mathcal{T}_{p,\alpha}$.

Theorem (D. Suárez, M. Mitkovski and BDW)

The Toeplitz algebra $\mathcal{T}_{p,\alpha}$ is equal to the closed algebra generated by $\{T_a : a \in \mathcal{A}\}$.

A Uniform Algebra and its Maximal Ideal Space

- For an element $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$ choose a net $z_\omega \to x$.
- Form S_{z_ω} and look at the limit operator obtained when $z_\omega \to x$, denote it by S_x.

Lemma (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{L}(A^p_\alpha)$. Then $B(S)(z) \to 0$ as $|z| \to 1$ if and only if $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$.

We can extend this to compute the essential norm of an operator S in terms of S_x where $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$.

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{T}_{p,\alpha}$. Then there exists a constant $C(p, \alpha, n)$ such that

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \|S_x\|_{\mathcal{L}(A^p_\alpha)} \approx \|S\|_e .$$

Proof of Main Theorem

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$ and $S \in \mathcal{L}(A^p_\alpha)$. Then S is compact if and only if $S \in \mathcal{T}_{p,\alpha}$ and $\lim_{|z|\to 1} B(S)(z) = 0$.

Proof.

$\Rightarrow$: If S is compact that $B(S)(z) \to 0$ as $|z| \to 1$ and $S \in \mathcal{T}_{p,\alpha}$.
$\Leftarrow$: If $S \in \mathcal{T}_{p,\alpha}$, then we have

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \|S_x\|_{\mathcal{L}(A^p_\alpha)} \approx \|S\|_e .$$

If $B(S)(z) \to 0$ as $|z| \to 1$, then $S_x = 0$ for all $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$.
This gives $\|S\|_e = 0$ or equivalently S is compact. □

The Hilbert Space Case

Theorem (D. Suárez, M. Mitkovski and BDW)

For $S \in \mathcal{T}_{2,\alpha}$ *we have*

$$\|S\|_e = \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \|S_x\|_{\mathcal{L}(A^2_\alpha)}$$

and

$$\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} r(S_x) \leq \lim_{k \to \infty} \left(\sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \left\| S_x^k \right\|_{\mathcal{L}(A^2_\alpha)}^{\frac{1}{k}} \right) = r_e(S)$$

with equality when S *is essentially normal.*

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $\alpha > -1$ *and* $S \in \mathcal{T}_{2,\alpha}$. *Then*

$$\|S\|_e = \sup_{\|f\|_{A^2_\alpha} = 1} \limsup_{|z| \to 1} \|S_z f\|_{A^2_\alpha}.$$

The Hilbert Space Case

Theorem (D. Suárez, M. Mitkovski and BDW)

Let $S \in \mathcal{T}_{2,\alpha}$. The following are equivalent:

(1) $\lambda \notin \sigma_e(S)$;

(2)

$$\lambda \notin \bigcup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \sigma(S_x) \quad \text{and} \quad \sup_{x \in M_{\mathcal{A}} \setminus \mathbb{B}_n} \left\| (S_x - \lambda I)^{-1} \right\|_{\mathcal{L}(A^2_\alpha)} < \infty;$$

(3) *There is a number $t > 0$ depending only on λ such that*

$$\|(S_x - \lambda I) f\|_{A^2_\alpha} \geq t \|f\|_{A^2_\alpha} \quad \text{and} \quad \left\| (S_x^* - \overline{\lambda} I) f \right\|_{A^2_\alpha} \geq t \|f\|_{A^2_\alpha}$$

for all $f \in A^2_\alpha$ and $x \in M_{\mathcal{A}} \setminus \mathbb{B}_n$.

Other Directions

The Fock space $\mathcal{F}$ is the collection of holomorphic functions f on $\mathbb{C}^n$ such that

$$\|f\|_{\mathcal{F}}^2 := \int_{\mathbb{C}^n} |f(z)|^2 \, e^{-\pi|z|^2} dv(z) < \infty.$$

This is a reproducing kernel Hilbert space with $k_\lambda(z) = e^{\pi z \overline{\lambda}}$ as kernel. Similar results are true for the Fock Space.

Theorem (W. Bauer and J. Isralowitz)

Let $S \in \mathcal{L}(\mathcal{F})$. Then S is compact if and only if $S \in \mathcal{T}_2$ and $\lim_{|z|\to\infty} B(S)(z) = 0$.

Corollary (W. Bauer and J. Isralowitz)

Let $S \in \mathcal{T}_2$, then

$$\|S\|_e \approx \sup_{\|f\|_{\mathcal{F}}=1} \limsup_{|z|\to\infty} \|S_z f\|_{\mathcal{F}}.$$

Other Directions

Let Ω be a bounded symmetric domain in $\mathbb{C}^n$. These are Hermitian symmetric spaces with a complete Riemannian metric given by the Bergman metric. For each $a \in \Omega$ there is a biholomorphic automorphism φ_a that interchanges 0 and a.

Let $A^2(\Omega)$ denote the Bergman space of analytic functions on Ω that are square integrable with respect to volume measure. This space has a reproducing kernel K_a, and relates to the automorphisms by

$$K_w(z) = K_{\varphi(w)}(\varphi(z)) J_c\varphi(w) \overline{J_c\varphi(w)}$$

Proposition (M. Mitkovski and BDW)

Let $S \in \mathcal{T}_2$, then

$$\|S\|_e \approx \sup_{\|f\|_{A^2(\Omega)}=1} \limsup_{z \to \partial\Omega} \|S_z f\|_{A^2(\Omega)}.$$

Open Questions

If $S \in \mathcal{L}(A^2_\alpha)$ then $B(S) \in L^\infty$. Similarly, if $S \in \mathcal{K}(A^2_\alpha)$ then $B(S) \to 0$ as $|z| \to 1$. And, even better, $S \in \mathcal{K}(A^2_\alpha)$ if and only if $S \in \mathcal{T}_{2,\alpha}$ and $B(S) \to 0$ as $|z| \to 1$.

Question

Can we characterize the Schatten class operators on A^2_α as those that belong to the Toeplitz algebra $\mathcal{T}_{2,\alpha}$ and an integrability condition on the Berezin transform $B(S)(z)$?

One can show that if $S \in \mathcal{S}_p$ then

$$\|B(S)\|_{L^p(\mathbb{B}_n;\lambda_n)} := \left(\int_{\mathbb{B}_n} |B(S)(z)|^p \, d\lambda_n(z)\right)^{\frac{1}{p}} \lesssim \|S\|_{\mathcal{S}_p}.$$

Proposition (M. Mitkovski and BDW)

Let $1 < p < \infty$ and $\alpha > -1$. If $S \in \mathcal{S}_p$, then $B(S) \in L^p(\mathbb{B}_n; \lambda_n)$ and $S \in \mathcal{T}_{2,\alpha}$.

Thank You!