# Bilinear Forms on the Dirichlet Space


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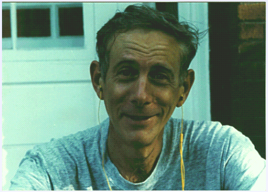


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# Talk Outline

- Motivation and History of the Problem
  - Review of Dirichlet Space Theory
  - Hankel Forms on the Dirichlet Space
- Main Result and Sketch of Proof
- Corollaries, Further Results and Questions

# Bilinear Forms on the Hardy Space

- The Hardy space $H^2(\mathbb{D})$ is the collection of all analytic functions on the disc such that

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^2 dm(\xi) < \infty$$

- The Hankel Operator $H_b$ maps $H^2(\mathbb{D})$ to $H^2(\mathbb{D})^\perp$ and is given by

$$H_b := (I - \mathbb{P}_{H^2}) M_b$$

- To study the boundedness of this operator, we can study only the corresponding bilinear Hankel form $T_b : H^2(\mathbb{D}) \times H^2(\mathbb{D}) \to \mathbb{C}$,

$$T_b(f,g) := \langle fg, b \rangle_{H^2}$$

# Bilinear Forms on the Hardy Space

- The bilinear form $T_b$ is bounded if and only if $b$ belongs to $BMOA(\mathbb{D})$.
- We can connect this to Carleson measures for the space $H^2(\mathbb{D})$.

**Lemma**

*A function $b \in BMOA(\mathbb{D})$ if and only if $b \in H^2(\mathbb{D})$ and*

$$|b'(z)|^2(1-|z|^2)dA(z)$$

*is a Carleson measure for $H^2(\mathbb{D})$.*

**Theorem**

*The bilinear form $T_b : H^2(\mathbb{D}) \times H^2(\mathbb{D}) \to \mathbb{C}$ is bounded if and only if*

$$|b'(z)|^2(1-|z|^2)dA(z)$$

*is a Carleson measure for $H^2(\mathbb{D})$.*

# The Dirichlet Space $\mathcal{D}^2(\mathbb{D})$

## Definition (Dirichlet Space)

An analytic function is an element of the Dirichlet space $\mathcal{D}^2(\mathbb{D})$ if and only if

$$\|f\|_{\mathcal{D}^2}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty$$

In terms of Fourier coefficients one has the equivalent norm given by

$$\|f\|_{\mathcal{D}^2}^2 \approx \sum_{n=0}^{\infty} \sqrt{n^2+1}|\hat{f}(n)|^2$$

The following inclusion relations hold

$$\mathcal{D}^2(\mathbb{D}) \subset H^2(\mathbb{D}) \subset A^2(\mathbb{D})$$

# Carleson Measures for $\mathcal{D}^2(\mathbb{D})$

## Definition

A measure $\mu$ on $\mathbb{D}$ is a $\mathcal{D}^2(\mathbb{D})$-Carleson measure if and only if

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C(\mu)^2 \|f\|_{\mathcal{D}^2}^2$$

for all $f \in \mathcal{D}^2(\mathbb{D})$.

- The best constant in the above embedding is called the norm of the Carleson measure
$$C(\mu) := \|\mu\|_{\mathcal{D}^2-\text{Carleson}}.$$
- This is a function theoretic quantity.
- We also want a geometric quantity that we can use to study Carleson measures.

# Carleson Measures for $\mathcal{D}^2(\mathbb{D})$

## Logarithmic Capacity on the Disc

For an interval $I \subset \mathbb{T}$, let $T(I)$ be the Carleson tent over the interval $I$,

$$T(I) := \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| \leq 1, \frac{z}{|z|} \in I \right\}$$

This definition obviously extends to general compact sets $E \subset \mathbb{T}$.
Given a compact subset $E \subset \mathbb{T}$, the capacity of the set $E$ is defined by

$$\text{cap}(E) := \inf \left\{ \|f\|_{\mathcal{D}^2}^2 : \text{Re}\, f \geq 1 \text{ on } T(E) \right\}.$$

It is easy to see that for an interval $I \subset \mathbb{T}$ we have

$$\text{cap}(I) \approx \left( \log \left( \frac{2\pi}{|I|} \right) \right)^{-1}$$

# Carleson Measures for $\mathcal{D}^2(\mathbb{D})$: Geometric Characertization

There is an obvious necessary condition a $\mathcal{D}^2(\mathbb{D})$-Carleson must satisfy: Suppose that $\mu$ is a $\mathcal{D}^2$-Carleson measure. For $\lambda \in \mathbb{D}$, let

$$k_\lambda(z) := 1 + \log \frac{1}{1 - \overline{\lambda} z}.$$

Then $k_\lambda \in \mathcal{D}^2(\mathbb{D})$ and $\|k_\lambda\|_{\mathcal{D}^2}^2 \approx -\log\left(1 - |\lambda|^2\right)$. Let $\tilde{k}_\lambda$ denote the (approximately) normalized version of $k_\lambda$.

For each interval $I \subset \mathbb{T}$ there exists a unique $\lambda \in \mathbb{D}$ with $1 - |\lambda|^2 = |I|$. Standard estimates show:

$$\frac{\mu\left(T(I)\right)}{\operatorname{cap}(I)} \lesssim \int_{\mathbb{D}} |\tilde{k}_\lambda(z)|^2 d\mu(z) \leq C(\mu)^2 \|\tilde{k}_\lambda\|_{\mathcal{D}^2}^2 \approx C(\mu)^2.$$

Analogue of the Carleson measure condition for $H^2(\mathbb{D})$. Unfortunately, this simple condition is not sufficient.

# Carleson Measures for $\mathcal{D}^2(\mathbb{D})$: Geometric Characertization

**Theorem (Stegenga (1980))**

*A measure $\mu$ is a Carleson measure for $\mathcal{D}^2(\mathbb{D})$ if and only if*

$$\mu\left(\cup_{j=1}^{N} T(I_j)\right) \leq S(\mu)\,\mathrm{cap}\left(\cup_{j=1}^{N} I_j\right),$$

*for all finite unions of disjoint arcs on the boundary $\mathbb{T}$.*

- This is a geometric characterization of the Carleson measures.
- But, it is difficult to check:
  - Computing capacity is hard.
  - One has to check every possible collection of disjoint intervals in $\mathbb{T}$.
- One has an equivalence between the quantities $C(\mu)$ and $S(\mu)$,

$$C(\mu)^2 \approx S(\mu).$$

# Hankel Operators on the Dirichlet Space

In an analogous manner, one defines the (small) Hankel operator $h_b : \mathcal{D}^2(\mathbb{D}) \to \overline{\mathcal{D}^2(\mathbb{D})}$ by

$$h_b := \overline{\mathbb{P}_{\mathcal{D}^2} M_b} = \int_{\mathbb{D}} \overline{b'(z)} f'(z) g(z) dA(z)$$

## Definition

Suppose that $b$ is analytic on $\mathbb{D}$. It belongs to $\mathcal{X}(\mathbb{D})$ if and only if

$$\int_{\mathbb{D}} |f(z)|^2 |b'(z)|^2 dA(z) \leq C^2 \|f\|_{\mathcal{D}^2}^2, \quad \forall f \in \mathcal{D}^2(\mathbb{D}). \qquad (\dagger)$$

Moreover,

$$\|b\|_{\mathcal{X}} := \inf\{C : (\dagger) \text{ holds}\} + |b(0)|$$

Namely, $d\mu_b(z) := |b'(z)|^2 dA(z)$ is a $\mathcal{D}^2(\mathbb{D})$-Carleson measure and

$$\|b\|_{\mathcal{X}} = \|\mu_b\|_{\mathcal{D}^2-\text{Carleson}} + |b(0)|.$$

# Hankel Operators on the Dirichlet Space

## Theorem (Rochberg, Wu (1993))

*Suppose that* $b$ *is analytic on* $\mathbb{D}$. *Then* $h_b$ *is bounded if and only if* $b \in \mathcal{X}(\mathbb{D})$. *Moreover,*

$$\|h_b\|_{\mathcal{D}^2 \to \overline{\mathcal{D}^2}} \approx \|\mu_b\|_{\mathcal{D}^2-\text{Carleson}}.$$

One can also look at the corresponding problem for the bilinear form $T_b : \mathcal{D}^2(\mathbb{D}) \times \mathcal{D}^2(\mathbb{D}) \to \mathbb{C}$. But, one can easily observe that the operator $h_b$ doesn't induce the bilinear form $T_b$.

## Conjecture

*The bilinear form* $T_b : \mathcal{D}^2(\mathbb{D}) \times \mathcal{D}^2(\mathbb{D}) \to \mathbb{C}$ *is bounded if and only if* $b \in \mathcal{X}(\mathbb{D})$.

# Main Result

**Theorem (N. Arcozzi, R. Rochberg, E. Sawyer, BDW (2008))**

*Let* $T_b : \mathcal{D}^2(\mathbb{D}) \times \mathcal{D}^2(\mathbb{D}) \to \mathbb{C}$ *be the bilinear form defined by*

$$\begin{aligned} T_b(f,g) &:= \langle fg, b\rangle_{\mathcal{D}^2} \\ &= f(0)g(0)\overline{b}(0) + \int_{\mathbb{D}} \overline{b'(z)}\left(f'(z)g(z) + f(z)g'(z)\right) dA(z). \end{aligned}$$

*Let* $d\mu_b(z) := |b'(z)|^2 dA(z)$. *Then* $T_b$ *is a bounded bilinear form on* $\mathcal{D}^2(\mathbb{D}) \times \mathcal{D}^2(\mathbb{D})$ *if and only if* $b \in \mathcal{X}(\mathbb{D})$ *with*

$$\|b\|_{\mathcal{X}} := \|\mu_b\|_{\mathcal{D}^2-\text{Carleson}} + |b(0)| \approx \|T_b\|_{\mathcal{D}^2\times\mathcal{D}^2\to\mathbb{C}}.$$

This Theorem demonstrates that the corresponding picture for the Hardy space $H^2(\mathbb{D})$ carries over to $\mathcal{D}^2(\mathbb{D})$.

# Carleson Measure $\Rightarrow$ Bounded Bilinear Form

Suppose that $\mu_b$ is a $\mathcal{D}^2(\mathbb{D})$-Carleson measure.
For $f, g \in \mathrm{Pol}\,(\mathbb{D})$ we have

$$T_b(f,g) := f(0)g(0)\overline{b}(0) + \int_{\mathbb{D}} \overline{b'(z)}\left(f'(z)g(z) + f(z)g'(z)\right) dA(z)$$

$$\begin{aligned} |T_b(f,g)| \quad &\leq \quad |f(0)g(0)\overline{b(0)}| + \int_{\mathbb{D}} |f'(z)g(z)\overline{b'(z)}| dA(z) \\ &+ \int_{\mathbb{D}} |f(z)g'(z)\overline{b'(z)}| dA(z) \end{aligned}$$

# Carleson Measure ⇒ Bounded Bilinear Form

$$
\begin{aligned}
|T_b(f,g)| &\leq |f(0)g(0)\overline{b}(0)| + \|f\|_{\mathcal{D}^2}\left(\int_{\mathbb{D}}|g(z)|^2 d\mu_b(z)\right)^{\frac{1}{2}} \\
&\quad + \|g\|_{\mathcal{D}^2}\left(\int_{\mathbb{D}}|f(z)|^2 d\mu_b(z)\right)^{\frac{1}{2}} \\
&\leq \left(|b(0)| + \|\mu_b\|_{\mathcal{D}^2-\text{Carleson}}\right)\|f\|_{\mathcal{D}^2}\|g\|_{\mathcal{D}^2} \\
&= \|b\|_{\mathcal{X}}\|f\|_{\mathcal{D}^2}\|g\|_{\mathcal{D}^2}.
\end{aligned}
$$

So, $T_b$ has a bounded extension from $\mathcal{D}^2(\mathbb{D}) \times \mathcal{D}^2(\mathbb{D}) \to \mathbb{C}$ with

$$\|T_b\|_{\mathcal{D}^2\times\mathcal{D}^2\to\mathbb{C}} \lesssim \|b\|_{\mathcal{X}}.$$

# Bounded Bilinear Form $\implies$ Carleson Measure

Choose an (almost) extremal collection of intervals $\{I_j\}_j \subset \mathbb{T}$ so that we have

$$S(\mu_b) := \sup \frac{\mu_b\left(\cup_{j=1}^N T(I_j)\right)}{\text{cap}(\cup_{j=1}^N I_j)} = \frac{\mu_b\left(\cup_{j=1}^N T(I_j)\right)}{\text{cap}(\cup_{j=1}^N I_j)}$$

This method of proof was suggested to us by Michael Lacey.
We will use this collection of intervals to construct functions $f$ and $g$ to test in the bilinear for $T_b$. We will prove an estimate of the form:

$$\frac{\mu_b\left(\cup_{j=1}^N T(I_j)\right)}{\text{cap}(\cup_{j=1}^N I_j)} \lesssim \|T_b\|^2_{\mathcal{D}^2\times\mathcal{D}^2\to\mathbb{C}}.$$

The function $g$ will be constructed using an approximate extremal function from the collection of intervals that achieves the supremum and will be approximately equal to the indicator function on $\cup_{j=1}^N T(I_j)$.
The function $f$ will be, approximately, $b'$ on the set $\cup_{j=1}^N T(I_j)$.

# Bounded Bilinear Form $\implies$ Carleson Measure

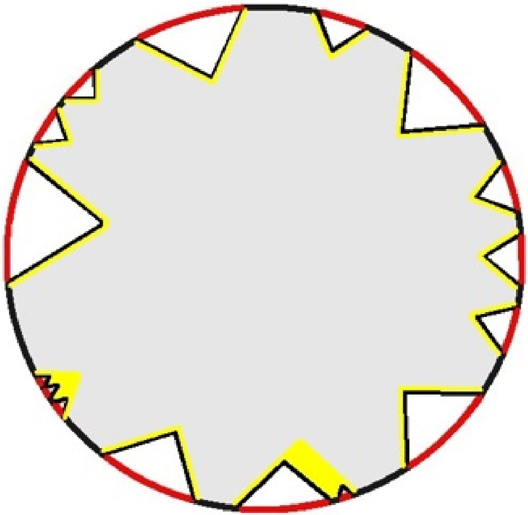
## Trees on $\mathbb{D}$

- There is a discrete version of the Dirichlet space that can be used simple model.
- To construct the dyadic tree $\mathcal{T}$, first form the Whitney decomposition. The center of each box is a vertex on the tree.
- The origin of $\mathbb{D}$ is the root of the tree, $o$. We say that a vertex $\alpha$ is a child of $\beta$ if the arc on $\mathbb{T}$ corresponding to $\alpha$ is a child of the arc corresponding to $\beta$.
- One then defines the dyadic Dirichlet Space as

$$B_2(\mathcal{T}) := \{f : \mathcal{T} \to \mathbb{C} : |f(o)|^2 + \sum_{\alpha \in \mathcal{T}} |\Delta f(\alpha)|^2 := \|f\|_{B_2}^2 < \infty\}$$

- One can recover results on $\mathcal{D}^2(\mathbb{D})$ from results on $B_2(\mathcal{T})$ by averaging, mean value properties, etc.
- The theory of the dyadic Dirichlet spaces and the connection with $\mathcal{D}^2(\mathbb{D})$ has been deeply explored by Arcozzi, Rochberg and Sawyer.

# Bounded Bilinear Form $\implies$ Carleson Measure



# Bounded Bilinear Form $\implies$ Carleson Measure

Using the dyadic tree $\mathcal{T}$, and the extremal intervals we selected, we can form a holomorphic function $\varphi$ that is basically the indicator of $\cup_j T(I_j)$.

## Lemma

*There exists a holomorphic function $\varphi$ such that*

$$\begin{cases} |\varphi(z) - \varphi(w_k^\alpha)| & \lesssim & \mathrm{cap}(\cup_{j=1}^N I_j), & z \in T(I_k^\alpha) \\ \mathrm{Re}\,\varphi(w_k^\alpha) & \geq & c > 0, & 1 \leq k \leq M_\alpha \\ |\varphi(w_k^\alpha)| & \leq & C, & 1 \leq k \leq M_\alpha \\ |\varphi(z)| & \lesssim & \mathrm{cap}(\cup_{j=1}^N I_j), & z \notin \cup_{j=1}^N T\left(I_j^\gamma\right) \end{cases}.$$

*Moreover,*

$$\|\varphi\|_{\mathcal{D}^2}^2 \lesssim \mathrm{cap}\left(\cup_{j=1}^N I_j\right).$$

# Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

We will use $g = \varphi^2$ and

$$f(z) := \int_{\cup_{j=1}^N T(I_j)} \frac{b'(\zeta)}{(1-\overline{\zeta}z)} \frac{dA(\zeta)}{\overline{\zeta}}$$

Using the reproducing kernel property we find that

$$\begin{aligned} f'(z) &= \int_{\cup_{j=1}^N T(I_j)} \frac{b'(\zeta)}{(1-\overline{\zeta}z)^2} dA(\zeta) \\ &= b'(z) - \int_{\mathbb{D}\setminus\cup_{j=1}^N T(I_j)} \frac{b'(\zeta)}{(1-\overline{\zeta}z)^2} dA(\zeta) \\ &=: b'(z) + \Lambda b'(z) \end{aligned}$$

This function $f$ is approximately $b'$ on the set $\cup_{j=1}^N T(I_j)$

# Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

If we substitute these into the bilinear form $T_b$ we find that:

$$\begin{aligned}
T_b(f,g) &= T_b(f,\varphi^2) = T_b(f\varphi,\varphi) \\
&= \int_{\mathbb{D}} \left\{ f'(z)\varphi(z) + 2f(z)\varphi'(z) \right\} \varphi(z)\overline{b'(z)}dA(z) \\
&\quad + f(0)\varphi(0)^2\overline{b(0)} \\
&= f(0)\varphi(0)^2\overline{b(0)} + \int_{\mathbb{D}} |b'(z)|^2\varphi(z)^2 dA(z) \\
&\quad + 2\int_{\mathbb{D}} \varphi(z)\varphi'(z)f(z)\overline{b'(z)}dA(z) + \int_{\mathbb{D}} \Lambda b'(z)\overline{b'(z)}\varphi(z)^2 dA(z) \\
&:= (1) + (2) + (3) + (4).
\end{aligned}$$

# Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

Term (1) is trivial.
Term (2) yields (using properties of $\varphi$ and a key geometric property) that

$$\begin{aligned}(2) &= \int_{\mathbb{D}} |b'(z)|^2\varphi(z)^2 dA(z)\\ &= \left\{\int_{\cup_{j=1}^N T(I_j)} + \int_{\cup_{j=1}^N T(I_j^\beta)\setminus\cup_{j=1}^N T(I_j)} + \int_{\mathbb{D}\setminus\cup_{j=1}^N T(I_j^\beta)}\right\}|b'(z)|^2\varphi(z)^2 dA\\ &=: (2_A)+(2_B)+(2_C).\end{aligned}$$

The main term $(2_A)$ satisfies

$$\begin{aligned}(2_A) &= \mu_b\left(\cup_{j=1}^N T(I_j)\right) + \int_{\cup_{j=1}^N T(I_j)} |b'(z)|^2\left(\varphi(z)^2-1\right) dA(z)\\ &= \mu_b\left(\cup_{j=1}^N T(I_j)\right) + O\left(\|T_b\|^2 \operatorname{cap}\left(\cup_{j=1}^N T(I_j)\right)\right).\end{aligned}$$

Terms $(2_B)$ and $(2_C)$ are error terms and are controlled by properties of $\varphi$.

# Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

Terms (3) and (4) are error terms.
Using properties of $\varphi$, geometric estimates, and Schur's Lemma, we can show these are controlled by estimates of the form

$$\epsilon\mu_b\left(\cup_{j=1}^n T(I_j)\right) + C(\epsilon)\|T_b\|^2_{\mathcal{D}^2\times\mathcal{D}^2\to\mathbb{C}}\,\mathrm{cap}\left(\cup_{j=1}^N I_j\right)$$

where $\epsilon > 0$ is a small number to be chosen later.
Thus, we have

$$\mu_b\left(\cup_{j=1}^n T(I_j)\right) \lesssim \epsilon\mu_b\left(\cup_{j=1}^n T(I_j)\right) + C(\epsilon)\|T_b\|^2_{\mathcal{D}^2\times\mathcal{D}^2\to\mathbb{C}}\,\mathrm{cap}\left(\cup_{j=1}^N I_j\right)$$

Choosing $\epsilon > 0$ small enough gives the result.

# Bounded Bilinear Form $\implies$ Carleson Measure

## Key Observations

If $T_b$ extends to a bounded bilinear form on $\mathcal{D}^2(\mathbb{D}) \times \mathcal{D}^2(\mathbb{D}) \to \mathbb{C}$ then $b \in \mathcal{D}^2(\mathbb{D})$. Setting $g = 1$ we obtain:

$$|\langle f, b\rangle_{\mathcal{D}^2}| = |T_b(f,1)| \leq \|T_b\|_{\mathcal{D}^2\times\mathcal{D}^2\to\mathbb{C}}\|f\|_{\mathcal{D}^2}$$

for all polynomials $f \in \mathrm{Pol}(\mathbb{D})$.This implies that $b \in \mathcal{D}^2(\mathbb{D})$ and

$$\|b\|_{\mathcal{D}^2} \lesssim \|T_b\|_{\mathcal{D}^2\times\mathcal{D}^2\to\mathbb{C}}.$$

### Proposition

*Given $\varepsilon > 0$ we can find $\beta = \beta(\varepsilon) < 1$ so that*

$$\mu_b\left(\cup_{j=1}^{N(\beta)} T(I_j^\beta) \setminus \cup_{j=1}^{N} T(I_j)\right) \leq \varepsilon \mu_b\left(\cup_{j=1}^{N} T(I_j)\right)$$

# Bounded Bilinear Form $\Longrightarrow$ Carleson Measure

## Proof.

Note that

$$\begin{aligned}\mu_b\left(\cup_{j=1}^{N} T(I_j)\right)+\mu_b\left(\cup_{j=1}^{N(\beta)} T(I_j^{\beta})\setminus\cup_{j=1}^{N} T(I_j)\right) &= \mu_b\left(\cup_{j=1}^{N(\beta)} T(I_j^{\beta})\right)\\ &\leq S(\mu_b)\,\mathrm{cap}\left(\cup_{j=1}^{N(\beta)} I_j^{\beta}\right).\end{aligned}$$

Since $\{I_j\}_{j=1}^N$ is the maximal collection of intervals in the geometric definition of $\mathcal{D}^2(\mathbb{D})$-Carleson measures. Next observe that

$$\mathrm{cap}\left(\cup_{j=1}^{N(\beta)} I_j\right)\leq(1+\varepsilon)\,\mathrm{cap}\left(\cup_{j=1}^{N} I_j\right).$$

This immediately gives that

$$\mu_b\left(\cup_{j=1}^{N(\beta)} T(I_j^{\beta})\setminus\cup_{j=1}^{N} T(I_j)\right)\leq\varepsilon\,\mathrm{cap}\left(\cup_{j=1}^{N} I_j\right).$$

# Corollaries of The Main Result

There is a close connection between boundedness of the bilinear form, duality theorems for function spaces, and weak factorization results.

## Definition

The weakly factored space $\mathcal{D}^2(\mathbb{D})\hat{\odot}\mathcal{D}^2(\mathbb{D})$ is the completion of finite sums $h = \sum f_j g_j$ under the norm

$$\|h\|_{\mathcal{D}^2\hat{\odot}\mathcal{D}^2} = \inf\left\{\sum \|f_j\|_{\mathcal{D}^2}\|g_j\|_{\mathcal{D}^2} : h = \sum f_j g_j\right\}.$$

## Corollary

*With the pairing* $(h, b) = \langle h, b\rangle_{\mathcal{D}^2} = T_b(h, 1)$ *we have that*

$$(\mathcal{D}^2(\mathbb{D})\hat{\odot}\mathcal{D}^2(\mathbb{D}))^* = \mathcal{X}(\mathbb{D}).$$

Namely, for $\Lambda \in \left(\mathcal{D}^2(\mathbb{D})\hat{\odot}\mathcal{D}^2(\mathbb{D})\right)^*$, there is a unique $b \in \mathcal{X}$ with $\Lambda h = T_b(h, 1)$ for $h \in \mathrm{Pol}(\mathbb{D})$, and $\|\Lambda\| = \|T_b\|_{\mathcal{D}^2\times\mathcal{D}^2\to\mathbb{C}} \approx \|b\|_{\mathcal{X}}$.

# Further Results and Questions

**Further Results:**

- Without much difficulty one can characterize the symbols of bounded bilinear forms that are bounded on $\mathcal{B}^p_\alpha(\mathbb{D}) \times \mathcal{B}^q_{-\alpha}(\mathbb{D}) \to \mathbb{C}$.
- One should be able to use these ideas to prove the corresponding inequality on the unit ball in $\mathbb{C}^n$. (Details still need to be checked!)

**Questions:**

- Can one give an intrinsic characterization of the space $\mathcal{X}(\mathbb{D})$? Equivalent question: Can one give an intrinsic characterization of the space $\mathcal{D}^2(\mathbb{D})\hat{\odot}\mathcal{D}^2(\mathbb{D})$?
- Can one prove the corresponding bilinear inequality for the spaces $\mathcal{D}^p(\mathbb{D}) \times \mathcal{D}^q(\mathbb{D})$?
- Can one prove the corresponding bilinear inequality for the spaces $\mathcal{D}^2_\alpha(\mathbb{D}) \times \mathcal{D}^2_\alpha(\mathbb{D}) \to \mathbb{C}$?

# Thank You!