Multi-Parameter Riesz Commutators

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Hilbert and Riesz Transforms

- The Hilbert Transform is defined by

$$H(f)(x) := \frac{1}{\pi}\int_{\mathbb{R}} f(y)\frac{1}{x-y}dy = f * \left(\frac{1}{\pi y}\right)(x).$$

- Which can be viewed on the Fourier Transform side as:

$$\widehat{H(f)}(\xi) := -i\text{sgn}(\xi)\hat{f}(\xi).$$

- The Riesz Transforms are the n-dimensional generalizations of the Hilbert Transform. For each $1 \leq j \leq n$ we have

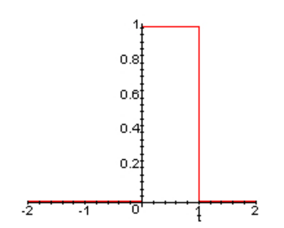
$$R_j(f)(x) := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^n} f(y)\frac{x_j - y_j}{|x-y|^{n+1}}dy = f * \left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\frac{y_j}{|y|^{n+1}}\right)(x).$$

- On the frequency side:

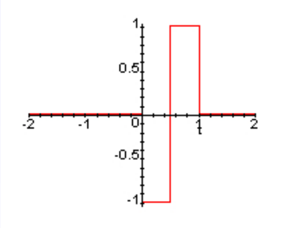
$$\widehat{R_j(f)}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi).$$

A Wavelet Basis for $L^2(\mathbb{R}^n)$

- Let $h^1(x) := \mathbf{1}_{[0,1)}(x)$ and let $h^0(x) := -\mathbf{1}_{[0,1/2)}(x) + \mathbf{1}_{[1/2,1)}(x)$



$h^1(x)$



$h^0(x)$

- Let

$$\mathcal{D}_n := \{2^{-k}(j + [0,1)^n) : j \in \mathbb{Z}^n, k \in \mathbb{Z}\}$$

i.e., the usual dyadic grid in $\mathbb{R}^n$.

A Wavelet Basis for $L^2(\mathbb{R}^n)$

- Let $\mathrm{Tr}_y(f)(x) := f(x-y)$ and $\mathrm{Dil}_t(f)(x) := t^{-n/2} f(\frac{x}{t})$.
- Define

$$\mathrm{Sig}^n := \{\epsilon = (\epsilon_1, \ldots, \epsilon_n) : \epsilon_i \in \{0,1\}\} \setminus \{(1, \ldots, 1)\}.$$

- For $Q \in \mathcal{D}_n$ and $\epsilon \in \mathrm{Sig}^n$ set

$$h_Q^\epsilon(x) := \prod_{j=1}^{n} \mathrm{Tr}_{c(Q)} \mathrm{Dil}_{|Q|} h^{\epsilon_j}(x_j).$$

- $\{h_Q^\epsilon : Q \in \mathcal{D}_n, \epsilon \in \mathrm{Sig}^n\}$ is the Haar wavelet basis for $L^2(\mathbb{R}^n)$.

The Space $BMO(\mathbb{R}^n)$

Definition

$$\|b\|_{BMO} := \sup_{Q} \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 dx$$

Theorem (C. Fefferman (1971))

The dual of $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$, i.e., $\left(H^1(\mathbb{R}^n)\right)^* = BMO(\mathbb{R}^n)$.

Definition (Square Function Characterization)

A function is in (dyadic) $BMO(\mathbb{R}^n)$ if and only if for any (dyadic) cube Q' we have a constant C such that:

$$\frac{1}{|Q'|} \sum_{Q \subset Q'} \sum_{\epsilon \in \mathrm{Sig}^n} |\langle b, h_Q^{\epsilon} \rangle|^2 \leq C.$$

BMO and Riesz Transforms

For each $j = 1, \dots, n$ define the following commutator operator on $L^2(\mathbb{R}^n)$:

$$[b, R_j](f)(x) := b(x)R_j(f)(x) - R_j(bf)(x).$$

Theorem (Coifman, Rochberg, and Weiss (1976))

Let $b \in BMO(\mathbb{R}^n)$, then for $j = 1, \dots, n$

$$\|[b, R_j]\|_{2\to 2} \lesssim \|b\|_{BMO(\mathbb{R}^n)}.$$

If $\|[b, R_j]\|_{2\to 2} < +\infty$ for $j = 1, \dots, n$, then

$$\|b\|_{BMO(\mathbb{R}^n)} \lesssim \max \|[b, R_j]\|_{2\to 2}.$$

Gives $BMO(\mathbb{R}^n)$ as an "operator space".

Product Spaces

- We are concerned with product spaces:

$$\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_t} = \otimes_{s=1}^{t} \mathbb{R}^{n_s}$$

- $\mathcal{D}^{\vec{n}} := \otimes_{s=1}^{t} \mathcal{D}_{n_s}$ is the tensor product of the usual dyadic grids in $\mathbb{R}^{n_s}$. Any $R \in \mathcal{D}^{\vec{n}}$ is of the form

$$R = Q_1 \otimes \cdots \otimes Q_t$$

  with each Q_s a dyadic cube in $\mathbb{R}^{n_s}$.
  Also, let $\mathrm{Sig}^{\vec{n}} := \{\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_t) : \epsilon_s \in \mathrm{Sig}^{n_s}\}$

Tensor Product Wavelet Basis in $L^2(\otimes_{s=1}^t \mathbb{R}^{n_s})$

- Take the Haar wavelet basis described earlier in $\mathbb{R}^{n_s}$, i.e.,

$$\{h_{Q_s}^{\epsilon_s} : Q_s \in \mathcal{D}_{n_s}, \epsilon_s \in \mathrm{Sig}^{n_s}\}$$

For each $R \in \mathcal{D}^{\vec{n}}$ and $\vec{\epsilon} \in \mathrm{Sig}^{\vec{n}}$ define the following function:

$$h_R^{\vec{\epsilon}}(x_1, \ldots, x_t) := \prod_{s=1}^{t} h_{Q_s}^{\epsilon_s}(x_s)$$

- $\{h_R^{\vec{\epsilon}} : R \in \mathcal{D}^{\vec{n}}, \vec{\epsilon} \in \mathrm{Sig}^{\vec{n}}\}$ is a wavelet basis for $L^2(\otimes_{s=1}^t \mathbb{R}^{n_s})$.

Product $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$

A Reasonable Guess:

Product BMO?

A function is in $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$ if and only if for any rectangle S in $\otimes_{s=1}^{t}\mathbb{R}^{n_s}$ there exists a constant C such that:

$$\frac{1}{|S|}\sum_{R\subset S}\sum_{\vec{\epsilon}\in \mathrm{Sig}^{\vec{n}}} |\langle b, h_R^{\vec{\epsilon}}\rangle|^2 \leq C$$

THIS IS WRONG!!!

Defines a space called "Rectangular" BMO, which is larger than product $BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$. (Counter-example do to Carleson).
Instead of rectangles, one must use arbitrary open sets in $\otimes_{s=1}^{t}\mathbb{R}^{n_s}$.

Product $BMO(\otimes_{s=1}^t \mathbb{R}^{n_s})$

Correct Definition:

Definition (Product BMO)

A function b is in $BMO(\otimes_{s=1}^t \mathbb{R}^{n_s})$ if and only if for any **open** set U in $\otimes_{s=1}^t \mathbb{R}^{n_s}$ with finite measure there exists a constant C such that:

$$\frac{1}{|U|} \sum_{R \subset U} \sum_{\vec{\epsilon} \in \mathrm{Sig}^{\vec{n}}} |\langle b, h_R^{\vec{\epsilon}} \rangle|^2 \leq C.$$

How do you check on every open set?

Theorem (S.-Y.A. Chang, R. Fefferman (1980))

The dual of product $H^1(\otimes_{s=1}^t \mathbb{R}^{n_s})$ is product $BMO(\otimes_{s=1}^t \mathbb{R}^{n_s})$, i.e., $\left(H^1(\otimes_{s=1}^t \mathbb{R}^{n_s})\right)^* = BMO(\otimes_{s=1}^t \mathbb{R}^{n_s})$.

$BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$ and Iterated Commutators

- Additional cancellation is present in the multi-parameter setting and this can still be studied via commutators.
- We need iterated (nested) commutators:
  Let R_{s,j_s} denote the j_s th Riesz transform taken in the s parameter variable.
  For $s = 1, \ldots, t$ and for $1 \leq j_s \leq n_s$ we consider the following iterated (nested) commutators on $L^2(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$:

$$[\cdots[b, R_{1,j_1}], R_{2,j_2}], \cdots], R_{t,j_t}](f)(x)$$

2 Parameter Iterated Commutator in $\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$

For $s = 1, 2$ and $1 \leq j_s \leq n_s$ the iterated commutator is:

$$\begin{aligned}[[b, R_{1,j_1}], R_{2,j_2}](f)(x) \quad := \quad & b(x)R_{1,j_1}R_{2,j_2}(f)(x) - R_{1,j_1}(b)(x)R_{2,j_2}(f)(x) \\ & -R_{2,j_2}(b)(x)R_{1,j_1}(f)(x) + R_{1,j_1}R_{2,j_2}(bf)(x)\end{aligned}$$

$BMO(\otimes_{s=1}^{t}\mathbb{R})$ as an Operator Space

Theorem (C. Sadosky and S. Ferguson (2001))

Let $b \in BMO(\otimes_{s=1}^{t}\mathbb{R})$, *then*

$$\|[\cdots[b, H_1], H_2], \cdots], H_t]\|_{2\to 2} \lesssim \|b\|_{BMO(\otimes_{s=1}^{t}\mathbb{R})}.$$

Theorem (M. Lacey and S. Ferguson (2002), M. Lacey and E. Terwilleger (2004))

If $\|[\cdots[b, H_1], H_2], \cdots], H_t]\|_{2\to 2} < +\infty$, *then*

$$\|b\|_{BMO(\otimes_{s=1}^{t}\mathbb{R})} \lesssim \|[\cdots[b, H_1], H_2], \cdots], H_t]\|_{2\to 2}.$$

Restatement of Nehari's Theorem for little Hankels on the polydisc.
KEY POINT: Provides a useful characterization of $BMO(\otimes_{s=1}^{t}\mathbb{R})$.

Main Result

It is possible to generalize the Coifman, Rochberg, Weiss result to the product setting:

Theorem (S. Petermichl, J. Pipher, M. Lacey, BW)

Let $b \in BMO(\otimes_{s=1}^{t} \mathbb{R}^{n_s})$*, then for* $s = 1, \ldots, t$*, and all* $1 \leq j_s \leq n_s$

$$\|[\cdots[b, R_{1,j_1}], R_{2,j_2}], \cdots], R_{t,j_t}]\|_{2\to 2} \lesssim \|b\|_{BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})}.$$

If $\|[\cdots[b, R_{1,j_1}], R_{2,j_2}], \cdots], R_{t,j_t}]\|_{2\to 2} < +\infty$ *for all* $s = 1, \ldots, t$ *and all* $1 \leq j_s \leq n_s$*, then*

$$\|b\|_{BMO(\otimes_{s=1}^{t}\mathbb{R}^{n_s})} \lesssim \max \|[\cdots[b, R_{1,j_1}], R_{2,j_2}], \cdots], R_{t,j_t}]\|_{2\to 2}.$$

Riesz Transforms and Dyadic Shifts

- The Riesz transforms can be recovered by an averaging of certain operators which map Haar functions to themselves (Haar shifts).
- For the dyadic grid $\mathcal{D}$ in $\mathbb{R}^n$ let $\sigma : \mathcal{D} \to \mathcal{D}$ with $2^n|\sigma(Q)| = |Q|$.
- Use the same notation for a map $\sigma : \mathrm{Sig}^n \to \mathrm{Sig}^n$.
- Let

$$\text{Ш} h_Q^{\varepsilon} := h_{\sigma(Q)}^{\sigma(\varepsilon)}.$$

Theorem (S. Petermichl, S. Treil, A. Volberg (2002))

- *The operator* Ш *is a bounded linear operator on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$.
- *The Riesz transforms are in the convex hull of the operators* Ш*, the convex hull taken with respect to the strong operator topology.*

Reduction to Commutators with Haar Shifts

We construct the Haar shifts Ш_s defined on $L^2(\mathbb{R}^{n_s})$ for each $s = 1, \ldots, t$.

Proposition

The operator

$$\vec{\text{Ш}} := \text{Ш}_1 \otimes \cdots \otimes \text{Ш}_t$$

extends to a bounded linear operator on $L^p(\mathbb{R}^{\vec{n}})$ *for all* $1 < p < \infty$.

To prove the upper bound in our theorem, it is sufficient to deduce the estimate for the operators:

$$\mathsf{C}_{\vec{\text{Ш}}}(b, f) := [\cdots[b, \text{Ш}_1], \cdots], \text{Ш}_t](f)$$

viewed as acting on $L^2(\mathbb{R}^{\vec{n}})$.

Multi-Parameter Paraproducts

Consider the bilinear operators, (multi-parameter paraproducts):

$$\Pi(f_1, f_2) := \sum_{R \in \mathcal{D}^{\vec{n}}} \epsilon_R \langle f_1, h_R^{\vec{\varepsilon}_1} \rangle \langle f_2, h_R^{\vec{\varepsilon}_2} \rangle \frac{h_R^{\vec{\varepsilon}_3}}{\sqrt{|R|}}.$$

Theorem (J.-L. Journé (1985), C. Muscalu, J. Pipher, T. Tao, and C. Thiele (2003), M. Lacey and J. Metcalfe (2004))

If for all $1 \le s \le t$, there is at most one choice of $j = 1, 2, 3$ with $\varepsilon_{j,s} = \vec{1}$, then the operator B *satisfies*

$$\Pi : L^p \times L^q \longrightarrow L^r, \qquad 1 < p, q < \infty, \ \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

If in addition, $\vec{\varepsilon}_1 \neq \vec{1}$, we will have the estimates

$$\Pi : BMO \times L^p \to L^p, \qquad 1 < p < \infty.$$

Main Idea in the Proof of the Upper Bound

We consider the one-parameter setting first:

$$C_{\text{Ш}}(b,f) := [b,\text{Ш}](f) = \sum_{Q,Q'\in\mathcal{D}} \sum_{\varepsilon,\varepsilon'\neq\vec{1}} \langle b, h_{Q'}^{\varepsilon'}\rangle \langle f, h_Q^{\varepsilon}\rangle [h_{Q'}^{\varepsilon'}\text{Ш}] h_Q^{\varepsilon}.$$

Compute the following:

$$[h_{Q'}^{\varepsilon'},\text{Ш}]h_Q^{\varepsilon}$$

$$[h_{Q'}^{\varepsilon'},\text{Ш}]h_Q^{\varepsilon} = \begin{cases} 0 & Q\cap Q'\neq\emptyset,\ \ Q\subsetneq Q' \\ \pm|Q|^{-1/2}h_{\sigma(Q)}^{\sigma(\varepsilon)} - \text{Ш}h_Q^{\epsilon'}h_Q^{\epsilon} & Q=Q' \\ |Q|^{-1/2}\left(\pm h_{\sigma(Q)}^{\varepsilon'} \pm h_{\sigma^2(Q)}^{\sigma(\varepsilon')}\right) & Q'=\sigma(Q) \\ \pm|Q|^{-1/2}h_{\sigma(Q')}^{\sigma(\varepsilon')} & 2^n|Q'|=Q,\ \ Q'\neq\sigma(Q) \\ |Q|^{-1/2}\left(\pm h_{Q'}^{\varepsilon'} \pm h_{\sigma(Q')}^{\sigma(\varepsilon')}\right) & 2^n|Q'|<|Q|. \end{cases}$$

Main Idea in the Proof of the Upper Bound

The computation demonstrates the following:

- The first line captures the essential cancellation in BMO and commutators.
- $\mathsf{C}_{Ш}(b, f)$ is a finite linear combination of terms of the form

$$Ш\Pi(b, f), \qquad \Pi(b, Шf)$$

  for appropriate choices of $Ш$ and paraproducts Π.
- These are good paraproducts. We can apply the previous theorem, and $\mathsf{C}_{Ш}(b, f)$ will be bounded on $L^2(\mathbb{R}^n)$ with norm controlled by $BMO(\mathbb{R}^n)$. This in turn implies $\mathsf{C}(b, f)$ is bounded.

Proof of the Upper Bound in the Multi-Parameter Setting

- To prove the upper bound in the multi-parameter setting, we "tensor" the previous argument.
- For the operators the Haar shifts Ш_s, we compute directly

$$[\cdots[h_R^{\vec{\varepsilon}}, \text{Ш}_1], \cdots], \text{Ш}_t]h_{R'}^{\vec{\varepsilon}'}$$

- The result is a tensor product of the one-parameter answer.
- We can write the commutator $\mathrm{C}_{\vec{\text{Ш}}}(b, f)$ as a finite linear combination of terms

$$\vec{\text{Ш}}\Pi(b, f)\,, \qquad \Pi(b, \vec{\text{Ш}} f)$$

  for different choices of multi-parameter paraproduct Π and different choices of operator $\vec{\text{Ш}}$.
- $\mathrm{C}_{\vec{\text{Ш}}}(b, f)$ will be bounded on $L^2(\mathbb{R}^{\vec{n}})$ with norm controlled by $BMO(\otimes_{s=1}^t \mathbb{R}^{n_s})$. Gives $\mathrm{C}(b, f)$ bounded with norm controlled by product BMO.

The Lower Bound

- We replace the Haar wavelet with the Meyer wavelet.
- Define a space reduced BMO, which plays the role of rectangle BMO. This space is "related" to product BMO via Journé's Lemma.
- If the commutators are bounded, then we have an initial weak lower bound in terms of reduced BMO. We want to boot-strap this lower bound to a lower bound in terms of product BMO.
- There are difficulties:
  - The approach used in Lacey-Ferguson and Lacey-Terwilleger depends upon the relationship between the Hilbert transform and projections.
  - We need to do something similar in the Hilbert transform case. To accomplish this we perform a reduction to deal with "nice" multipliers.
  - With this reduction it is possible to implement the general scheme established in the papers Lacey-Ferguson and Lacey-Terwilleger.

Other Problems Considered

- The theorem also implies a weak factorization result for the product Hardy space $H^1(\otimes_{s=1}^{t}\mathbb{R}^{n_s})$ in terms of L^2 functions and Riesz transforms.
- Commutators in One-Parameter have connections to Div-Curl Lemmas.

  Let E be a divergence free vector field, and B be a curl free vector field, then

  $$E \cdot B \in H^1(\mathbb{R}^n)$$

  Our theorem implies a new Div-Curl Lemma, but one which allows divergence/curl free vector fields in each variable separately.

  Connections with partial differential equations.