Math 2401
Exam 1

Name:______________________________

By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:______________________________

Problem	Possible	Earned
1	10	
2	15	
3	10	
4	10	
5	10	
6	5	
Total	60	

1. (10 pts) Evaluate the arclength of the curve

$$\mathbf{r}(t) = \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}, t \right)$$

from the values $0 \leq t \leq 1$.

$$r'(t) = \left(\frac{e^t - e^{-t}}{2}, \frac{e^t + e^{-t}}{2}, 1 \right)$$

$$\Rightarrow \| r'(t) \|^2 = 1 + \left(\frac{e^t - e^{-t}}{2} \right)^2 + \left(\frac{e^t + e^{-t}}{2} \right)^2$$

$$= 1 + \frac{1}{4} e^{2t} + \frac{1}{4} e^{-2t} + \cancel{\frac{1}{4} \cdot (-2)}$$

$$+ \frac{1}{4} e^{2t} + \frac{1}{4} e^{-2t} + \cancel{\frac{1}{4} \cdot 2}$$

$$= 1 + \frac{1}{2} e^{2t} + \frac{1}{2} e^{-2t} = \frac{1}{2} \left(2 + e^{2t} + e^{-2t} \right)$$

$$= \frac{1}{2} \left(e^t + e^{-t} \right)^2$$

$$\Rightarrow \| r'(t) \| = \frac{\sqrt{2}}{2} \cdot \left(e^t + e^{-t} \right)$$

$$\text{Arc Length} = \int_0^1 \| r'(t) \| \, dt$$

$$= \int_0^1 \frac{\sqrt{2}}{2} \left(e^t + e^{-t} \right) dt$$

$$= \frac{\sqrt{2}}{2} \left(e^t - e^{-t} \right) \Big|_0^1 = \boxed{\frac{\sqrt{2}}{2} \left(e - e^{-1} \right)}$$

2. (10 pts) Calculate the second-order partial derivatives of

$$g(x,y) = 2x^2y + e^{xy}.$$

$$\partial_x g = 4xy + ye^{xy}$$

$$\partial_y g = 2x^2 + xe^{xy}$$

$$\boxed{\begin{aligned} \partial_x^2 g &= 4y + y^2 e^{xy} \\ \partial_y^2 g &= x^2 e^{xy} \\ \partial_x \partial_y g &= 4x + e^{xy} + xye^{xy} \\ \partial_y \partial_x g &= 4x + e^{xy} + xye^{xy} \end{aligned}}$$

3. (15 pts) Set

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}.$$

Evaluate the limits of $f(x, y)$ as $(x, y) \to (0, 0)$:

(a) along the x-axis;

(b) along the y-axis;

(b) along the line $y = mx$;

(c) along the parabola $y = ax^2$.

Using these computation, does

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4 + y^2} \text{ exist?}$$

(a) X-axis $(X, 0)$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = 0} \qquad y \equiv 0$$

(b) y-axis $(0, y)$ i.e., $X \equiv 0$ so same as (a)

(c) Along $y = mx$ $\quad f(x, mx) = \dfrac{2x^2(mx)}{x^4 + m^2x^2} = \dfrac{2mx^3}{x^2(m^2+x^2)}$

$$= \frac{2mx}{m^2+x^2}$$

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{2mx}{m^2+x^2} = 0 = \lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} f(x,y)$$

(d) $\displaystyle\lim_{x\to 0} f(x, ax^2) = \lim_{x\to 0} \frac{2x^2(ax^2)}{x^4 + a^2x^4} = \lim_{x\to 0} \frac{2ax^4}{x^4(1+a^2)} = \frac{2a}{1+a^2} \neq 0.$

So, NO $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$ does not exist.

4. (10 pts) For $t > 0$, find the unit tangent and the principal normal of the function

$$\mathbf{r}(t) = \left(\cos t + t \sin t, \sin t - t \cos t, \frac{\sqrt{3}}{2} t^2\right).$$

$$r'(t) = \left(-\cancel{\sin t} + \cancel{\sin t} + t\cos t,\ \cancel{\cos t} - \cancel{\cos t} + t\sin t,\ \sqrt{3}\,t\right)$$

$$= (t\cos t,\ t\sin t,\ \sqrt{3}\,t)$$

$$\Rightarrow \|r'(t)\| = \sqrt{t^2 + 3t^2} = 2t \qquad (t>0)$$

$$\boxed{\hat{T}(t) = \left(\frac{\cos t}{2}, \frac{\sin t}{2}, \frac{\sqrt{3}}{2}\right)}$$

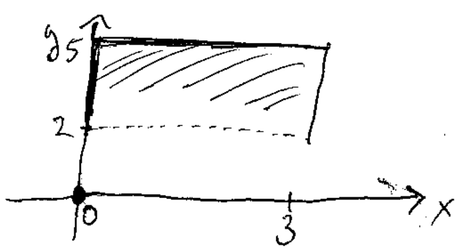
$$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|} = \frac{\left(-\frac{\sin t}{2}, \frac{\cos t}{2}, 0\right)}{\sqrt{\frac{1}{4}\sin^2 t + \frac{1}{4}\cos^2 t}} = \boxed{\begin{matrix}(-\sin t, \cos t, 0) \\ \parallel \\ \hat{N}(t)\end{matrix}}$$

5. (10 pts) Specify the interior and boundary of the set

$$\{(x, y) : 0 \leq x \leq 3, 2 < y \leq 5\}.$$

Sketch the set and determine if it is open, closed or neither.

Sketch:



Neither open nor closed

Interior $= \{(x,y) : 0 < x < 3,\ 2 < y < 5\}$

Boundary $= \{(x,2) : 0 \leq x \leq 3\} \cup \{(x,5) : 0 \leq x \leq 3\}$
$\cup \{(0,y) : 2 \leq y \leq 5\} \cup \{(3,y) : 2 \leq y \leq 5\}$

6. (5 pts) Evaluate the following quantities:

(a)

$$\int_0^2 2t\mathbf{i} + (t^2 - 1)\mathbf{j}dt$$

(b)

$$\lim_{t\to e^2}\left(t\ln t\mathbf{i} + \frac{\ln t}{t^2}\mathbf{j} + \sqrt{\ln t^2}\mathbf{k}\right)$$

(a) $\int_0^2 2t\hat{\imath} + (t^2-1)\hat{\jmath}\,dt$

$= \frac{t^2}{2}\Big|_0^2 \hat{\imath} + \frac{t^3}{2} - t\Big|_0^2 \hat{\jmath}$

$= 2\hat{\imath} + \left(\frac{8}{2} - 2\right)\hat{\jmath} = \boxed{2\hat{\imath} + 2\hat{\jmath}}$

(b) $\lim_{t\to e^2} t\ln t\,\hat{\imath} + \frac{\ln t}{t^2}\hat{\jmath} + \sqrt{\ln t^2}\,\hat{k}$

$\boxed{= 2e^2\hat{\imath} + \frac{2}{e^4}\hat{\jmath} + 2\hat{k} = \left(2e^2, \frac{2}{e^4}, 2\right).}$