Math 2401
Exam 2

Name: Key

By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:________________________

Problem	Possible	Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

1. (10 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of

$$f(x, y, z) = x + y + z$$

subject to the constraint that

$$x^2 + y^2 + z^2 = 1.$$

$g(x, y, z) = x^2 + y^2 + z^2 - 1$

$\nabla f = (1, 1, 1)$

$\nabla g = (2x, 2y, 2z)$

$\nabla f = \lambda \nabla g \implies 1 = 2\lambda x \quad 1 = 2\lambda y \quad 1 = 2\lambda z$

$\implies x = y = z$

Substitute in implies

$x^2 + x^2 + x^2 = 1 \implies 3x^2 = 1 \implies x = \pm\frac{1}{\sqrt{3}}$

Candidate Points: $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

Inspection gives Max: $\frac{3}{\sqrt{3}} = f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Min: $-\frac{3}{\sqrt{3}} = f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

2. (10 pts)

(a) Find an equation of the tangent plane to the surface

$$z = e^x \ln y$$

at $(3, 1, 0)$.

(b) Find the normal line to the surface at the point $(3, 1, 0)$.

Define $g(x,y,z) = z - e^x \ln y$

$\nabla g(x,y,z) = \left(-e^x \ln y, -\frac{e^x}{y}, 1\right)$

$\nabla g(3,1,0) = \left(-e^3 \ln 1, -\frac{e^3}{1}, 1\right) = (0, -e^3, 1).$

Normal Line:

$l(t) = (3,1,0) + t(0,-e^3,1)$

Tangent Plane: $\nabla g(3,1,0) \cdot (\hat{x} - (3,1,0)) = 0$

$= (0, -e^3, 1) \cdot (x-3, y-1, z) = 0$

$-e^3(y-1) + z = 0.$

3. (15 pts) Find the local maximum, minimum and saddle points of the following function:

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

$$\nabla f(x,y) = (4x^3 - 4y,\ 4y^3 - 4x)$$

$$\nabla f(x,y) = \hat{0} \iff 4x^3 = 4y \quad \text{and} \quad 4y^3 = 4x$$

$$x^3 = y \quad \text{and} \quad y^3 = x$$

$$\Rightarrow (y^3)^3 = y \quad \Rightarrow y^9 = y$$

$$\Rightarrow y(y^8 - 1) = 0 \qquad y = 0, 1, -1$$

Critical Points: $(0,0), (1,1), (-1,-1)$.

$$\text{Hessian}(f) = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$\text{Hessian}(f)(0,0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \Rightarrow \det < 0 \Rightarrow \text{Saddle}$$

$$\text{Hessian}(f)(1,1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \Rightarrow \det = 144 - 16 > 0 \Rightarrow \text{Min since } \partial_x^2 f > 0$$

$$\text{Hessian}(f)(-1,-1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \Rightarrow \det = 144 - 16 > 0 \Rightarrow$$

So at $(0,0)$ we have a saddle

$(1,1)$ we have local min.

$(-1,-1)$

4. (10 pts) Find the absolute maximum and minimum values of

$$g(x, y) = 2x^2 + x + y^2 - 2$$

on the set

$$\Omega = \{(x, y) : x^2 + y^2 \leq 4\}.$$

$\nabla g(x,y) = (4x - 1,\ 2y)$ $\qquad$ $\nabla g(x,y) = 0 \iff x = \frac{1}{4},\ y = 0$

$(\frac{1}{4}, 0) \in \Omega$

$\boxed{g(\frac{1}{4}, 0)} = 2\cdot(\frac{1}{4})^2 + \frac{1}{4} - 2$

$= \frac{2}{16} + \frac{4}{16} - 2 = \frac{6}{16} - \frac{32}{16}$

$= -\frac{26}{16}$

On boundary $x^2 + y^2 = 4$

$\Rightarrow y^2 = 4 - x^2$

$g(x,y)$ on boundary $= \phi(x) = 2x^2 + x + 4 - x^2 - 2$ $\qquad x \in [-2, 2]$.

$\phi(x) = x^2 + x - 2$

$\phi'(x) = 2x + 1$ $\qquad \phi'(x) = 0 \iff x = -\frac{1}{2}$ $\qquad (-\frac{1}{2}, \pm\sqrt{4 - \frac{1}{4}})$ Candidate

$(-\frac{1}{2}, \pm\frac{\sqrt{15}}{2})$.

$(\pm 2, 0)$.

$g(2,0) = 8 + 2 - 2 = 8$

$g(-2,0) = 8 - 2 - 2 = 4$

$g(-\frac{1}{2}, \frac{\sqrt{15}}{2}) = 2(-\frac{1}{2})^2 - \frac{1}{2} + \frac{15}{4} - 2 = \cancel{\frac{1}{2} - \frac{1}{2}} + \frac{15}{4} - 2 = \frac{7}{4}$

$g(-\frac{1}{2}, -\frac{\sqrt{15}}{2}) = 2(-\frac{1}{2})^2 - \frac{1}{2} + \frac{15}{4} - 2 = \frac{7}{4}$

Max at $(2,0)$ of 8

Min at $(\frac{1}{4}, 0)$ of $\frac{-26}{16}$

5. (10 pts) Evaluate the following quantities:

(a) Find the directional derivative of the function $f(x,y) = xy^2 + z^3$ at the point $(1,-2,1)$ in the direction $\mathbf{u} = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

(b) Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial s}$ if

$$z = xe^y + ye^{-x}, \quad x = e^t, \quad y = st^2.$$

(a) $\nabla f = (y^2, 2xy, 3z^2)$

$\nabla f(1,-2,1) = (4,-4,3)$

$\nabla f(1,-2,1) \cdot \hat{u} = (4,-4,3) \cdot (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{11}{\sqrt{3}}$

(b) $\frac{\partial z}{\partial s} = \nabla z \cdot \partial_s \hat{x}$

$=$

$\nabla z = (e^y - ye^{-x}, xe^y + e^{-x})$

$\partial_s \hat{x} = (0, t^2)$

$\partial_t \hat{x} = (e^t, 2st)$

$\nabla z(x(s,t), y(s,t)) = (e^{st^2} - st^2 e^{-e^t}, e^t e^{st^2} + e^{-e^t})$

$\frac{\partial z}{\partial s} = (e^{st^2} - st^2 e^{-e^t}, e^t e^{st^2} + e^{-e^t}) \cdot (0, t^2)$

$\frac{\partial z}{\partial t} = (e^{st^2} - st^2 e^{-e^\tau}, e^t e^{st^2} + e^{-e^t}) \cdot (e^t, 2st).$

6. (10 pts) Suppose that $\mathbf{F}(x,y) = (y - e^x \cos y, x + e^x \sin y)$.

(a) Is $\mathbf{F}(x,y)$ the gradient of some function $f(x,y)$?

(b) If so, find the function $f(x,y)$ so that $\nabla f(x,y) = \mathbf{F}(x,y)$.

(a) $F(x,y) = (P(x,y), Q(x,y))$

$$\frac{\partial P}{\partial y} = 1 + \sin y \cdot e^x$$

$$\frac{\partial Q}{\partial x} = 1 + e^x \sin y$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ So yes $\hat{F} = \nabla f$.

(b)

~~f(x,y)~~ $P(x,y) = \frac{\partial f}{\partial x}$

$$\Rightarrow f(x,y) = \int P(x,y)\,dx + \phi(y)$$

$$= xy - e^x \cos y + \phi(y)$$

$$\frac{\partial f}{\partial y} = x + e^x \sin y + \phi'(y) = Q(x,y)$$

$$\Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = C.$$

So $\boxed{f(x,y) = xy - e^x \cos y}$ works to give $\nabla f = \hat{F}$.