Math 2401
Exam 3

Name:______________________________

By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:______________________________

Problem	Possible	Earned
1	15	
2	10	
3	10	
4	10	
5	15	
Total	60	

1. Use Triple Integrals to evaluate the solid in 1^{st} Octant bounded above by $z = 4x^2 + 4y^2$, below by $z = -1$, on the sides by $y = x$, $y = x^2$.

Solution:

Note: This problem is poorly stated. 1^{st} Octant contradicts the value $z = -1$. We accepted either $\int_{z=0}^{4x^2+4y^2}$ or $\int_{z=-1}^{4x^2+4y^2}$.

$$V = \iiint_T dx\,dy\,dz$$

Note: $x^2 = x \iff x = 0, x = 1$

For $0 \le x \le 1$ we have $x^2 \le x$.

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} \int_{z=-1}^{4x^2+4y^2} dz\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{x^2}^{x} (4x^2 + 4y^2 + 1)\,dy\,dx$$

$$= \int_{x=0}^{1} 4x^2 y + \frac{4y^3}{3} + y \Big|_{x^2}^{x} dx =$$

$$= \int_{x=0}^{1} \left(4x^3 + \frac{4x^3}{3} + x - 4x^4 - \frac{4x^6}{3} - x^2\right) dx$$

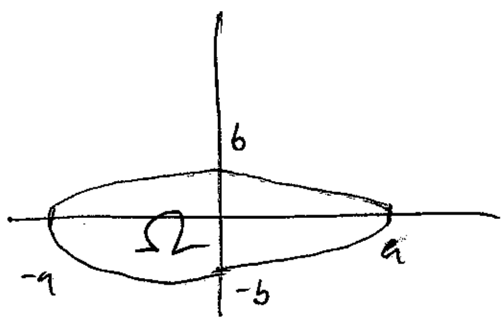
$$= x^4 + \frac{x^4}{3} + \frac{x^2}{2} - \frac{4}{5}x^5 - \frac{4x^7}{21} - \frac{x^3}{3} \Big|_0^1 = 1 + \frac{1}{3} + \frac{1}{2} - \frac{4}{5} - \frac{4}{21} - \frac{1}{3} = \frac{1563}{990}$$

2. Compute the area of the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2.$$

Sketch of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

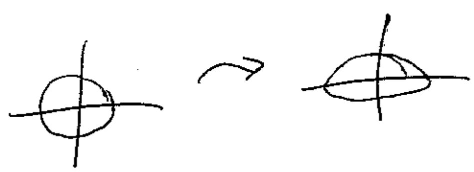


$$\text{Area (Ellipse)} = \iint_{\Omega} dx\,dy$$

$$= \int_{\{0<r<1\}} ab\, r\, dr\, d\theta$$

If we set $x(r,\theta) = a r\cos\theta$

$y(r,\theta) = b r\sin\theta$

This maps the circle  as $0 \le \theta \le 2\pi$, $0 \le r \le 1$.

$$\frac{\partial x}{\partial r} = a\cos\theta \qquad \frac{\partial y}{\partial r} = b\sin\theta \qquad \Rightarrow J(r,\theta) = ab\,r$$

$$\frac{\partial x}{\partial \theta} = -a r\sin\theta \qquad \frac{\partial y}{\partial \theta} = b r\cos\theta$$

$$ab\int_0^1\int_{\theta=0}^{2\pi} r\,dr\,d\theta = 2\pi\cdot ab\cdot\int_0^1 r\,dr = \frac{2\pi ab}{2}\, r^2\Big|_0^1 = \boxed{\pi ab}$$

3. Evaluate

$$\iiint_T \frac{1}{x^2+y^2+z^2}\, dx\,dy\,dz$$

T: $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$ $\quad 0 \le z \le \sqrt{1-x^2-y^2}$

Using cylindrical coordinates we have

$$\int_{\rho=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \frac{1}{\rho^2}\, \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

$0 \le \rho \le 1$ since $z^2 \le 1 - x^2 - y^2$

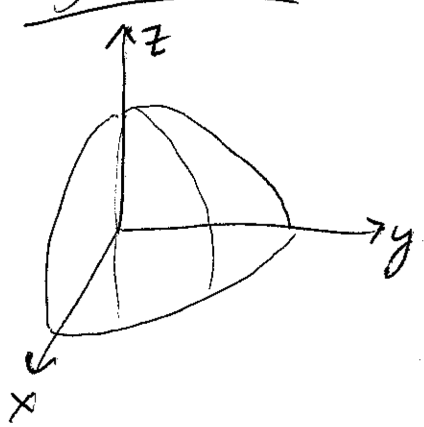
$0 \le \phi \le \frac{\pi}{2}$ since $0 \le z$

$0 \le \theta \le \frac{\pi}{2}$ since $x^2+y^2 \le 1$.

$0 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$

Sketch of T:



$$= \rho\Big|_0^1 \cdot \theta\Big|_0^{\pi/2} \cdot \left(-\cos\phi\right)\Big|_0^{\pi/2}$$

$$= 1 \cdot \frac{\pi}{2} \cdot 1 = \boxed{\frac{\pi}{2}}$$

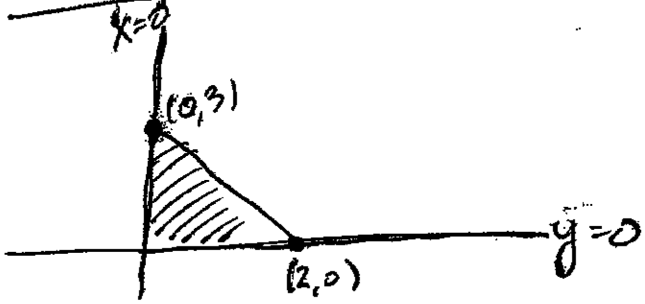
4. (10 pts) Suppose that Ω is the domain in the xy-plane given by the triangular region in the first quadrant bounded by $x = 0$, $y = 0$, $3x + 2y = 6$. If the mass density of Ω is given by $\lambda(x, y) = x + y$,

(a) Find the mass of Ω;

(b) Find the center of mass of of Ω.

Sketch of Ω:

$$3x+2y=6 \iff y = 3 - \frac{3}{2}x$$



$$\text{Mass}(\Omega) = \iint_\Omega \lambda(x,y)\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{3-\frac{3}{2}x} (x+y)\,dy\,dx$$

$$= \int_0^2 xy + \frac{y^2}{2}\Big|_0^{3-\frac{3}{2}x} dx$$

$$= \int_0^2 x\left(3-\frac{3}{2}x\right) + \frac{\left(3-\frac{3}{2}x\right)^2}{2}\,dx$$

$$= \int_0^2 \left(3-\frac{3}{2}x\right)\left(x + \frac{3}{2} - \frac{3}{4}x\right)dx$$

$$= \int_0^2 \left(3-\frac{3}{2}x\right)\left(\frac{3}{2} + \frac{1}{4}x\right)dx$$

$$= \frac{1}{4}\cdot\frac{1}{2}\int_0^2 (6-3x)(6+x)\,dx = 5$$

Center of Mass (x_M, y_M) $\quad x_M = \dfrac{\iint_\Omega x\,\lambda(x,y)\,dx\,dy}{\text{Mass}(\Omega)}$

$$x_M = \frac{1}{5}\int_0^2\int_0^{3-\frac{3}{2}x} x(x+y)\,dy\,dx \qquad y_M = \frac{1}{5}\int_0^2\int_0^{3-\frac{3}{2}x} y(x+y)\,dy\,dx$$

$$x_M = \frac{7}{10} \qquad y_M = \frac{6}{5}$$

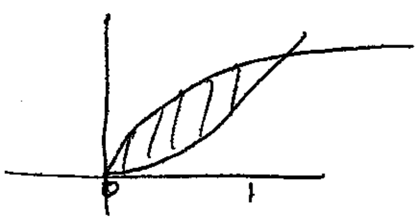
5. (15 pts) Sketch the regions of interest and evaluate the following quantities. If easier, change the order of integration, or the coordinates that you integrate with respect to.

(a) $\int_0^1 \int_{x^2}^{x^{1/4}} (x^{1/2} - y^2)dydx$

(b) $\int_0^1 \int_{x^2}^1 \frac{x^3}{\sqrt{x^4+y^2}}dydx$

(c) $\int_0^1 \int_0^{\sqrt{1-x^2}} \sin\sqrt{x^2+y^2}dydx$

(a)



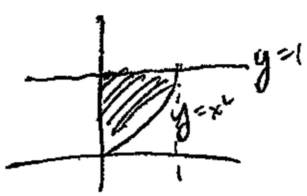
$$\int_0^1 \int_{x^2}^{x^{\frac14}} (x^{\frac12} - y^2)\,dy\,dx$$

$$= \int_0^1 x^{\frac12}y - \frac{y^3}{3}\Big|_{x^2}^{x^{\frac14}} dx$$

$$= \int_0^1 \left(x^{\frac34} - x^{\frac52} - \frac{x^{\frac34}}{3} + \frac{x^6}{3}\right)dx$$

$$= \int_0^1 \left(\frac{2}{3}x^{3/4} - x^{5/2} + \frac{x^6}{3}\right) =$$

(b)

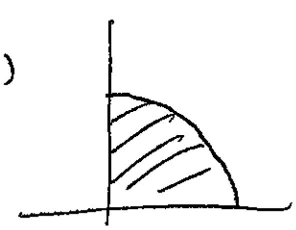


$$\int_0^1 \int_{x^2}^1 \frac{x^3}{\sqrt{x^4+y^2}}\,dy\,dx = \int_0^1 x^3\left(\int_{x^2}^1 \frac{dy}{\sqrt{y^2+x^4}}\right)dx$$

$$= \int_0^1 x^3 \left(\ln\left(y + \sqrt{x^4+y^2}\right)\right)\Big|_{x^2}^1$$

$$= \int_0^1 x^3\left(\ln x^2 - \ln\left(x^2 + \sqrt{2}x^2\right)\right)dx = \ln\left(\frac{1}{1+\sqrt{2}}\right)\cdot\frac14.$$

(c)



$$\int_0^1 \int_0^{\sqrt{1-\cancel{x^2}}} \sin\sqrt{x^2+y^2}\,dx\,dy$$

$$= \int_0^1 \int_0^{\pi/2} r\sin r\,dr\,d\theta = \frac{\pi}{2}\cdot\int_0^1 r\sin r\,dr$$

$$= \frac{\pi}{2}\left(-r\cos r + \sin r\right)\Big|_0^1 = \boxed{\frac{\pi}{2}\left(\sin 1 - \cos 1\right)}$$

$+ \quad r \quad \sin r$

$- \quad 1 \quad \cancel{\cos r}$

$0 \quad -\sin r$