1. 3 pts $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

4 pts for $f = \ldots$

3 pts correct answer

2. 4 pts for Green's Thm

3 pts for setting up line integral correctly

3 pts correct answer.

3. 5 pts $\iint_S d\sigma = \iint_\Omega \|\hat{N}(u,v)\|\, du\, dv$

5 pts correct $\|\hat{N}(u,v)\|$

5 pts $\left\langle \begin{array}{l} 3 \text{ Determining } \Omega \\ 2 \text{ correct answer} \end{array} \right.$

4. 3 pts $\text{Flux}_{\hat{V}} = \iint_S \hat{V} \cdot \hat{N}\, d\sigma$

4 pts for $\text{Flux} = \iiint_T \nabla \cdot \hat{V}\, dV$

3 pts for $\nabla \cdot \hat{V}$ correct

5 pts for correct answer

5. 3 pts $t=0, t=1$

4 pts $y^2\,dx + (x^2 - xy)\,dy$ in terms of t correct

3 pts Correct answer.

Math 2401
Exam 3

Name: Key

By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged: ______________________

Problem	Possible	Earned
1	10	
2	10	
3	15	
4	15	
5	10	
Total	60	

1. (10 pts) Calculate the line integral

$$\int_C (x^2 - y)dx + (y^2 - x)dy$$

where C is the path from $(0,0)$ to $(1,1)$ given by

$$\mathbf{r}(u) = \left(u^{2009}\cos\left(\frac{\pi}{2}(1-u)\right), \sin\left(u\frac{\pi}{2}\right)e^{(u^{1123}-1)}\right), \quad u \in [0,1].$$

Hint: Use the Fundamental Theorem of Line Integrals.

Note $P(x,y) = x^2 - y$ $\quad \frac{\partial P}{\partial y} = -1$

$Q(x,y) = y^2 - x$ $\quad \frac{\partial Q}{\partial x} = -1$

So the vector field is a gradient.

$f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - xy$ satisfies $\partial_x f = P$, $\partial_y f = Q$.

So $\boxed{\int_C (x^2-y)\,dx + (y^2-x)\,dy = f(1,1) - f(0,0) = \boxed{-\frac{1}{3}.}}$

2. (10 pts) Show that the area inside of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is πab. Hint: Use Green's Theorem or Jacobians.

By Green's Theorem

$$\iint_\Omega dx\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Let $\Omega = \{(x,y) : b^2x^2 + a^2y^2 \leq a^2b^2\}$. = Ellipse

~~The~~ The curve C is parameterized by

$x(t) = a\cos t$

$y(t) = b\sin t$

$dx = -a\sin t\,dt$

$dy = b\cos t\,dt$

$t \in [0, 2\pi]$.

$\Rightarrow x\,dy - y\,dx$

$= (ab\cos^2 t - ab\sin t(-\sin t))dt$

$= ab\,dt$

$$\frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{ab}{2}\cdot 2\pi = \boxed{\pi ab}$$

|| Green's Thm

$$\iint_\Omega dx\,dy = \text{Area of Elipse}$$

3. (15 pts) Calculate the surface area of the part of the cone $z = \sqrt{y^2 + x^2}$ that between the planes $z = 0$ and $z = 3$.

$$SA = \iint_S d\sigma$$

$$z = 0 \Leftrightarrow \sqrt{x^2+y^2} = 0$$

$$z = 3 \Leftrightarrow \sqrt{x^2+y^2} = 3$$

$$\Rightarrow r = 3$$

$$\Omega = \{(r,\theta) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}.$$

$$= \iint_\Omega \|\hat{N}(u,v)\| \, du\, dv$$

Here $\hat{r}(u,v) = (u, v, \sqrt{x^2+y^2}) \Rightarrow$

$$\|\hat{N}(u,v)\| = \sqrt{1 + \left(\tfrac{1}{2} 2x (x^2+y^2)^{-1/2}\right)^2 + \left(\tfrac{1}{2} 2y (x^2+y^2)^{-1/2}\right)^2}$$

$$= \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{1+1} = \sqrt{2}.$$

$$= \iint_\Omega \sqrt{2}\, du\, dv = \sqrt{2} \cdot \text{Area}(\Omega)$$

$$= \sqrt{2} \cdot 9\pi = 9\pi\sqrt{2}$$

4. (15 pts) Let S denote the surface given by the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$ (i.e., the boundary of the cube $0 \leq x \leq a$, $0 \leq y \leq a$ and $0 \leq z \leq a$). Calculate the flux of the vector field

$$\mathbf{v}(x, y, z) = (x^2, y^2, z^2)$$

through the surface S.

$$\text{Flux}_{\hat{V}}(S) = \iint_S \hat{V} \cdot \hat{N} \, d\sigma$$

$$= \iiint_T \nabla \cdot \hat{V} \, dV$$

$$\nabla \cdot \hat{V} = 2x + 2y + 2z$$

$$= \iiint_T (2x + 2y + 2z) \, dx\,dy\,dz$$

$$= a^2 \cdot \int_0^a 2x\,dx + a^2 \int_0^a 2y\,dy + a^2 \int_0^a 2z\,dz$$

$$= a^2 \cdot a^2 + a^2 \cdot a^2 + a^2 \cdot a^2$$

$$= 3a^2$$

$$\boxed{\text{Flux}_{\hat{V}}(S) = 3a^2.}$$

5. (10 pts) Evaluate

$$\int_C y^2 dx + (x^2 - xy) dy$$

where C is the path from $(0,0)$ to $(1,1)$ given by $\mathbf{r}(t) = (t, t^3)$.

Note

$x(t) = t$ $\qquad dx = dt$ $\qquad \hat{r}(0) = (0,0)$

$y(t) = t^3$ $\qquad dy = 3t^2$ $\qquad \hat{r}(1) = (1,1)$

$$y^2 dx + (x^2 - xy) dy = t^6 dt + (t^2 - t^4) 3t^2 dt$$

$$= (3t^4 - 2t^6) dt$$

$$\Rightarrow \int_C y^2 dx + (x^2 - xy) dy = \int_0^1 (3t^4 - 2t^6) dt$$

$$= \frac{3t^5}{5} - \frac{2t^7}{7} \Big|_0^1$$

$$= \frac{3}{5} - \frac{2}{7} = \frac{21 - 10}{35} = \boxed{\frac{11}{35}}.$$