Math 2605
Exam 3

Name:____________________________

By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:____________________________

Problem	Possible	Earned
1	15	
2	15	
3	15	
4	15	
Total	60	

1. (15 pts) Diagonalize the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

A is diagonal so

$$\mu_\pm = \frac{2-2}{2} \pm \sqrt{1 + \left(\frac{2-(-2)}{2}\right)^2}$$

$$= \pm\sqrt{1+2^2} = \pm\sqrt{5}$$

So $\mu_+ = \sqrt{5}$

$$\boxed{\text{Set } D = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{bmatrix}}$$

Consider $A - \mu_+ I_2 = \begin{pmatrix} 2-\sqrt{5} & 1 \\ 1 & -2-\sqrt{5} \end{pmatrix}$

$\hat{r}_1 = (2-\sqrt{5}, 1)$ $\quad \hat{r}_1^{\perp} = (-1, 2-\sqrt{5})$

Set $\hat{u}_1 = \dfrac{(-1, 2-\sqrt{5})}{\sqrt{1+4-4\sqrt{5}+5}} = \dfrac{(-1, 2-\sqrt{5})}{\sqrt{10-4\sqrt{5}}}$

Set $\hat{u}_2 = \dfrac{1}{\sqrt{10-4\sqrt{5}}}(2-\sqrt{5}, 1)$

Set $U = [u_1, u_2] = \begin{bmatrix} -1 & 2-\sqrt{5} \\ 2-\sqrt{5} & 1 \end{bmatrix} \cdot \dfrac{1}{\sqrt{10-4\sqrt{5}}}$

Then $D = U^T A U$ or $A = U^T A U$

4. (15 pts) Prove the following statements:

(a) Suppose that $A = A^T$. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A that correspond to eigenvalues μ and λ. Suppose that $\lambda \neq \mu$ and show that $\mathbf{u}$ is perpendicular to $\mathbf{v}$.

(b) Suppose that B is a $n \times m$ matrix. Prove that the eigenvalues of the matrix $B^T B$ are always non-negative.

(c) Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a collection of orthonormal vectors. Show that the collection $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

(a) If $\hat{u}, \hat{v}$ are e-vectors of A then

$A\hat{u} = \mu\hat{u} \quad A\hat{v} = \lambda\hat{v}$.

$\mu\langle\hat{u},\hat{v}\rangle = \langle\mu\hat{u},\hat{v}\rangle = \langle A\hat{u},\hat{v}\rangle = \langle\hat{u}, A^T\hat{v}\rangle$

$\lambda\langle\hat{u},\hat{v}\rangle = \langle\hat{u}, \lambda\hat{v}\rangle = \langle\hat{u}, A\hat{v}\rangle$

$\Rightarrow (\mu-\lambda)\langle\hat{u},\hat{v}\rangle = 0 \quad \mu \neq \lambda \Rightarrow \langle\hat{u},\hat{v}\rangle = 0 \Rightarrow \hat{u} \perp \hat{v}$.

(b) B^TB is a $m \times m$ matrix.

Suppose $(B^TB\hat{u}) = \lambda\hat{u} \Rightarrow \lambda\langle\hat{u},\hat{u}\rangle_{\mathbb{R}^m} = \langle B^TB\hat{u},\hat{u}\rangle_{\mathbb{R}^m}$

$= \langle B\hat{u}, B\hat{u}\rangle_{\mathbb{R}^n}$

$\Rightarrow \lambda \|\hat{u}\|^2_{\mathbb{R}^m} = \|B\hat{u}\|^2_{\mathbb{R}^n} \Rightarrow \lambda \geq \dfrac{\|B\hat{u}\|^2}{\|\hat{u}\|^2} \geq 0$.

Note $\|\hat{u}\| \neq 0$ since $\hat{u}$ is an e-vector

(c) $\{\hat{v}_1, \ldots, \hat{v}_k\}$ O.N. $\Rightarrow \hat{v}_j \perp \hat{v}_l \quad l \neq j \quad v_j \cdot v_j = 1$.

Suppose $\hat{0} = \alpha_1\hat{v}_1 + \cdots + \alpha_k\hat{v}_k$

$\Rightarrow 0 = \alpha_1\langle\hat{v}_1,\hat{v}_j\rangle + \cdots + \alpha_j\langle\hat{v}_j,\hat{v}_j\rangle + \cdots + \alpha_k\langle\hat{v}_k,\hat{v}_j\rangle$

$\Rightarrow \alpha_j = 0$ since all other inner products are zero $\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.