1. 3 pts Area $= \iint_{\Omega} dx\,dy$

4 pts for Jacobians or ~~corrections~~ correct eval.

3 pts correct answer.

2. 5 pts Correct Domain

5 pts Volume $= \iiint$ over correct domain

5 pts Correct answer

3. 3 pts $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$ for $dx\,dy\,dz$

4 pts Correct domain in spherical coordinates

3 pts Correct answer

4. 6 pts for domain of integration (circle in xy-plane)

3 pts $dV = r\,dr\,d\theta\,dz$

3 pts for correct triple integral

3 pts for correct answer

Math 2605
Exam 4

Name: Key

By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:____________________

Problem	Possible	Earned
1	10	
2	15	
3	10	
4	15	
Total	50	

1. (10 pts) Show that the area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is πab. Hint: Jacobians make this problem easy, though it can be done by hand.

Set $x(r,\theta) = ar\cos\theta$

$y(r,\theta) = br\sin\theta$

If $\Gamma = \{(r,\theta) : 0 \le r < 1,\ 0 \le \theta \le 2\pi\}$

$$\text{Area of Ellipse} = \iint_\Omega dx\,dy$$

$$= \iint_\Gamma |J(r,\theta)|\,dr\,d\theta$$

$$J(r,\theta) = \det\begin{pmatrix} a\cos\theta & b\sin\theta \\ -ar\sin\theta & br\cos\theta \end{pmatrix}$$

$$= abr(\cos^2\theta + \sin^2\theta) = abr$$

$$\text{Area of Ellipse} = \iint_\Gamma ab\,r\,dr\,d\theta = ab\cdot\left(\int_0^1 r\,dr\right)\cdot\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}\cdot 2\pi\cdot ab = \boxed{\pi ab}$$

Find the volume of the first-octant solid bounded by the planes $z=x$, $y-x=2$ and the cylinder $y=x^2$.

Sol$^{\underline{n}}$.

T is $\begin{cases} 0 \leq z \leq x \\ (x,y) \in \Omega_{xy} \text{ defined by } \cdot\ y = x+2 \text{ and } y=x^2. \\ \qquad\qquad\qquad\qquad\qquad\quad \cdot\ x \geq 0,\ y \geq 0. \end{cases}$

Points of intersection $x^2 = x+2 \Rightarrow x^2 - x - 2 = 0$
$\Rightarrow x = 2$ or $x = -1$

Since $x \geq 0$, we take $x=2$ only and therefore $0 \leq x \leq 2$

When $x = 1 \in [0,2]$: $\begin{cases} x^2 = 1 \\ x+2 = 3 \end{cases} \Rightarrow x+2 > x^2$

so $x^2 \leq y \leq x+2$

Hence T is $\cdot\ 0 \leq z \leq x$

$\left.\begin{array}{l} \cdot\ 0 \leq x \leq 2 \\ x^2 \leq y \leq x+2 \end{array}\right\} \Omega_{xy}$

and

$$vol(T) = \int_0^2 \left(\int_{x^2}^{x+2} \left(\int_0^x dz \right) dy \right) dx$$

$$= \int_0^2 \left(\int_{x^2}^{x+2} x \, dy \right) dx$$

$$= \int_0^2 x(x+2-x^2)\, dx$$

$$= \frac{x^3}{3} + x^2 - \frac{x^4}{4} \Big|_0^2$$

$$= \frac{8}{3} + 4 - 4 = \frac{8}{3}.$$

3. (10 pts) Evaluate

$$\iiint_T \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$$

where T is the region with $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, and $0 \le z \le \sqrt{1-x^2-y^2}$. Hint: Spherical Coordinates.

$$\rho^2 = x^2 + y^2 + z^2$$

$$\iiint_T \frac{1}{(x^2+y^2+z^2)}\,dx\,dy\,dz = \iiint \frac{\cancel{\rho^2}\sin\phi}{\cancel{\rho^2}}\,d\rho\,d\phi\,d\theta$$

$$0 \le z \le \sqrt{1-x^2-y^2} \Rightarrow 0 \le \rho \le 1.$$

$$\Rightarrow 0 \le \phi \le \frac{\pi}{2}$$

$$0 \le x \le 1$$
$$0 \le y \le \sqrt{1-x^2} \Rightarrow 0 \le \theta \le \frac{\pi}{2}$$

$$\rightarrow \int_{\theta=0}^{\pi/2} d\theta \cdot \int_{\rho=0}^{1} d\rho \cdot \int_{\phi=0}^{\pi/2} \sin\phi\,d\phi$$

$$= \frac{\pi}{2} \cdot 1 \cdot (-\cos\phi)\Big|_0^{\pi/2} = \boxed{\frac{\pi}{2}}.$$

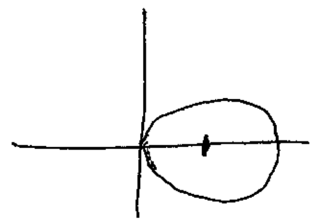
4. (15 pts) Find the volume of the solid bounded above by the cone $z^2 = x^2 + y^2$, below by the plane $z = 0$ and on the sides by the cylinder $x^2+y^2 = 2x$. Hint: Cylindrical Coordinates.

$$\text{Volume} = \iint_{\Omega} \left(\int_0^{\sqrt{x^2+y^2}} dz \right) dx\,dy$$

$$\sqrt{x^2+y^2} = r$$

$$x^2+y^2-2x=0 \iff (x^2-2x+1)+y^2=1$$

$$\iff (x-1)^2+y^2=1$$



$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \left(\int_{r=0}^{2\cos\theta} \left(\int_{z=0}^{r} dz \right) r\,dr \right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \left(\int_{r=0}^{2\cos\theta} r^2\,dr \right) d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} \frac{8\cos^3\theta}{3}\,d\theta$$

$u = \sin\theta \qquad du = \cos\theta\,d\theta$

$$= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\pi/2} \cos\theta\,(1-\sin^2\theta)\,d\theta$$

$u(-\frac{\pi}{2}) = -1$

$u(\frac{\pi}{2}) = 1$

$$= \frac{8}{3}\int_{-1}^{1} (1-u^2)\,du = \frac{8}{3}\left. u-\frac{u^3}{3}\right|_{-1}^{1} = \frac{8}{3}\left(1-\frac{1}{3}-(-1)+\frac{-1}{3}\right)$$

$$= \frac{8}{3}\left(2-\frac{2}{3}\right) = \frac{8\cdot 2\cdot 2}{3\cdot 3} = \frac{32}{9}$$