16(b).

Let A=the set of all limits of sequences of points of S that converge in E

First, to show that any element in A is also an element in $\bar{S}$

$\bar{S}$ is the intersection of closed sets => $\bar{S}$ is closed.

Using Theorem, because $\bar{S}$ is closed, the limit of any sequence of points of $\bar{S}$ that converges in E is in $\bar{S}$.

Since $S \subset \bar{S}$, any sequence of points in S is also a sequence of points of $\bar{S}$.

=> the limit of any sequence of points of S that converges in E is also in $\bar{S}$

=> $A \subset \bar{S}$ 2 *

Next, to show that any element in $\bar{S}$ is also in A

Proof by contradiction: Assume $p \in \bar{S}$ s.t. p is not limit of any sequence of points of S

Then, $\exists \epsilon > 0\ \forall s \in S$ s.t. $d(p,s) > \epsilon$

Then $\bar{S} - B_p(\epsilon)$ is also a closed subset of E that contains S

Need to Say $p \in \bar{S} \subset \bar{S} \setminus B_p(\varepsilon)$

But, $\bar{S}$ is in all closed subset of E that contains S, which is an contradiction

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=> $\bar{S} \subset A$

Thus, $\bar{S} = A$

Additional Problems:

> 1. Suppose that $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in $\mathbb{R}$. Suppose that $a_n \to L, c_n \to L$ and there exists an integer N such that if $n \geq N$, then
>
> $$a_n \leq b_n \leq c_n.$$
>
> Show that $b_n \to L$.

Given $\epsilon > 0, \exists\ M, n \geq M$ s.t.

$$L - \epsilon < a_n < L + \epsilon.$$

Given $\epsilon > 0, \exists\ K, n \geq K$ s.t.

$$L - \epsilon < c_n < L + \epsilon.$$

2 *

Take $Y = max\{M, K, N\}$. Then for a given $\epsilon > 0, \exists\ Y, n \geq Y$ s.t.

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon.$$

$$L - \epsilon < b_n < L + \epsilon.$$

$$|b_n - L| < \epsilon$$

Therefore, $b_n \to L$. ■

2) Suppose that $\{s_n\}$ is a sequence in $\mathbb{R}^n$. Show that if $s_n \to s$, then $||s_n|| \to ||s||$. Is the converse true?

$s_n \to s$ implies that $\exists N$ such that $d(s_n, s) < \epsilon$ for all $n > N$. For the same N, we have $|||s_n|| - ||s||| \leq ||s_n - s||$ (normed case of triangle inequality).

But $||s_n - s|| = d(s_n, s)$, which we have above that $d(s_n, s) < \epsilon$, thus $|||s_n|| - ||s||| < \epsilon$ for all $n > N$. This proves the statement. The converse is not necessarily. Take s_n to be the n-tuple $= ||s||$ with $s = (1, 1, \ldots,$ $((-1)^n, (-1)^n, \ldots, (-1)^n)$. We have $||s_n|| = \sqrt{n}$ for all n thus it converges to $\sqrt{n}$, but s_n itself can't possibly converge (it just oscillates between $(-1, -1, \ldots, -1)$ and $(1,1,\ldots,1)$).

4.

Let $\{a_n\}$ be a Cauchy sequence.

Let $\{a_{f(n)}\}$ be a subsequence of $\{a_n\}$, then we have $f(n) \geq n$ for all n

By definition of Cauchy sequence, $\forall \epsilon > 0$ $\exists N$ s.t $\forall m, n \geq N$, $d(a_m, a_n) < \epsilon$

Since $f(m) \geq m$ and $f(n) \geq n$ => $f(m), f(n) \geq N$ => $d(a_{f(m)}, a_{f(n)}) < \epsilon$

=> $\forall \epsilon > 0$ $\exists N$ s.t $\forall m, n \geq N$, $d(a_{f(m)}, a_{f(n)}) < \epsilon$

=> $\{a_{f(n)}\}$ is a Cauchy sequence

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③ If $S_n \to S$ then for every $\mathcal{E}$

$$|S_n - S| < \mathcal{E} \quad \text{if} \quad n \geq N.$$

$\frac{2}{2}$

Need to show:

$$|\sqrt{S_n} - \sqrt{S}| < \mathcal{E} \quad \text{for some } n \geq N.$$

Case 1: $S \neq 0$

Note that

$$|S_n - S| = |(\sqrt{S_n} - \sqrt{S})(\sqrt{S_n} + \sqrt{S})|$$

so $$|\sqrt{S_n} - \sqrt{S}| = \frac{|S_n - S|}{|\sqrt{S_n} + \sqrt{S}|}$$

Suppose we chose $\mathcal{E}$ as $\mathcal{E}\sqrt{S}$

then $$\frac{|S_n - S|}{|\sqrt{S_n} + \sqrt{S}|} \leq \frac{|S_n - S|}{|\sqrt{S}|}.$$

Now suppose $n \geq N$ s.t. $|S_N - S| < \mathcal{E}\sqrt{S}$

then $$\frac{|S_n - S|}{|\sqrt{S}|} < \frac{\mathcal{E}\sqrt{S}}{|\sqrt{S}|} = \mathcal{E}$$

So $\therefore$ $|\sqrt{S_n} - \sqrt{S}| < \mathcal{E}$ for some $n \geq N$

turn

Case 2: $S = 0$

The since $S_n \to S$

$|S_n - S| < \varepsilon^2$ for some $n \geq N$

$= |S_n - 0| < \varepsilon^2$

$= |S_n| < \varepsilon^2$

then $\sqrt{S_n} < \varepsilon$ for ~~all~~ $n \geq N$

in particular

$|\sqrt{S_n} - \sqrt{0}| < \varepsilon$

$= |\sqrt{S_n} - 0| < \varepsilon$ for $n \geq N$

which shows that $\sqrt{S_n} \to \sqrt{S}$

5. Two metrics d_1 and d_2 on a set X are *uniformly equivalent* if there exist constants $A > 0$ and $B > 0$ such that

$$d_1(x, y) \le A d_2(x, y) \;\; \forall x, y \in X$$

and

$$d_2(x, y) \le B d_1(x, y) \;\; \forall x, y \in X$$

Show that the metric space (X, d_1) is complete if and only if (X, d_2) is complete.

Answer:

Since the problem statement contains an "if and only if" statement, we must show two logical directions. First, let us show that (X, d_1) complete implies that (X, d_2) is complete.

Consider a Cauchy sequence $\{p_n\} \subset (X, d_2)$. We would like to show that $\{p_n\}$ is Cauchy in (X, d_1) which tells us that $\{p_n\}$ converges in (X, d_1) and we would then like to relate this convergence back to (X, d_2). To show that $\{p_n\}$ is Cauchy in (X, d_1), given any $\epsilon > 0$, we must find some N such that $m, n \ge N \Rightarrow d_1(p_m, p_n) < \epsilon$. To do this, we can use the fact that $\{p_n\}$ is Cauchy in (X, d_2). Here, find N such that $m, n \ge N$ implies that $d_2(p_m, p_n) < \frac{\epsilon}{A}$, where ϵ is given. Since we are given that $d_1(x, y) \le A d_2(x, y)$, we know that $\frac{d_1(p_m, p_n)}{A} \le d_2(p_m, p_n)$ which here means that $\frac{d_1(p_m, p_n)}{A} < \frac{\epsilon}{A}$ which in turn means that $d_1(p_m, p_n) < \epsilon, \forall m, n \ge N$. This means that $\{p_n\}$ is Cauchy in (X, d_1), and, since (X, d_1) is complete, that it converges in (X, d_1). This means that given any $\epsilon > 0$, $\exists N$ *s.t.* $n \ge N \Rightarrow d_1(p_n, p) < \epsilon$ where p is the limit point of $\{p_n\}$.

Now let us relative this convergence back to (X, d_2). Given some $\epsilon > 0$, find N such that $n \ge N$ implies $d_1(p, p_n) < \frac{\epsilon}{B}$. Since $d_2(x, y) \le B d_1(x, y)$ we know that $\frac{d_2(p, p_n)}{B} \le d_1(p, p_n)$ so that $\frac{d_2(p, p_n)}{B} < \frac{\epsilon}{B}$ which implies that $d_2(p, p_n) < \epsilon \;\forall n \ge N$. Thus we see that when (X, d_1) is complete, every Cauchy sequence in (X, d_2) converges, so that (X, d_1) complete $\Rightarrow (X, d_2)$ complete.

Now let us show that (X, d_2) complete implies that (X, d_1) is complete. Consider a Cauchy sequence $\{p_n\} \subset (X, d_1)$. We would like to show that $\{p_n\}$ is Cauchy in (X, d_2) which tells us that $\{p_n\}$ converges in (X, d_2) and we would then like to relate this convergence back to (X, d_1). To show that $\{p_n\}$ is Cauchy in (X, d_2), given any $\epsilon > 0$, we must find some N such that $m, n \ge N \Rightarrow d_2(p_m, p_n) < \epsilon$. To do this, we can use the fact that $\{p_n\}$ is Cauchy in (X, d_1). Here, find N such that $m, n \ge N$ implies that $d_1(p_m, p_n) < \frac{\epsilon}{B}$. Since we are given that $d_2(x, y) \le B d_1(x, y)$, we know that $\frac{d_2(p_m, p_n)}{B} \le d_1(p_m, p_n)$ which here means that $\frac{d_2(p_m, p_n)}{B} < \frac{\epsilon}{B}$ which in turn means that $d_2(p_m, p_n) < \epsilon, \forall m, n \ge N$. This means that $\{p_n\}$ is Cauchy in (X, d_2), and, since (X, d_2) is complete, that it converges in (X, d_2). This means that given any $\epsilon > 0$, $\exists N$ *s.t.* $n \ge N \Rightarrow d_2(p_n, p) < \epsilon$ where p is the limit point of $\{p_n\}$.

Now let us relative this convergence back to (X, d_1). Given some $\epsilon > 0$, find N such that $n \ge N$ implies $d_2(p, p_n) < \frac{\epsilon}{A}$. Since $d_1(x, y) \le A d_2(x, y)$ we know that $\frac{d_1(p, p_n)}{A} \le d_2(p, p_n)$ so that $\frac{d_1(p, p_n)}{A} < \frac{\epsilon}{A}$ which implies that $d_1(p, p_n) < \epsilon \;\forall n \ge N$. Thus we see that when (X, d_2) is complete, every Cauchy sequence in (X, d_1) converges, so that (X, d_2) complete $\Rightarrow (X, d_1)$ complete.

With both directions shown, the proof is complete.

6. Let (X, d) be a metric space, and let $\{q_n\} \subset X$ be a convergent sequence with limit q. Let $p \in X$ and then show that $d(p, q) = \lim_{n\to\infty} d(p, q_n)$.

To show that $d(p, q) = \lim_{n\to\infty} d(p, q_n)$, need to show that $\forall \epsilon > 0$ there exists an $N(\epsilon) > 0$ such that $\forall n \ge N(\epsilon)$, $|d(p, q) - d(p, q_n)| < \epsilon$.
First, $|d(p, q) - d(p, q_n)| = |d(q, p) - d(p, q_n)|$, then by the reverse triangle inequality $|d(q, p) - d(p, q_n)| \le d(q, q_n)$.
Since $q_n \to q$, $\forall \epsilon > 0$, $\exists N(\epsilon)$ s.t. $\forall n \ge N(\epsilon)$, $d(q, q_n) < \epsilon$); therefore $|d(p, q) - d(p, q_n)| \le d(q, q_n) < \epsilon$.
We have $\forall \epsilon > 0$, $\exists N(\epsilon)$ s.t. $\forall n \ge N(\epsilon)$, $|d(p, q) - d(p, q_n)| < \epsilon$); therefore $d(p, q) = \lim_{n\to\infty} d(p, q_n)$.

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