III.18 Prove that $\underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n$, with equality holding if and only if the sequence converges.

Show that $\underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n$.
By the definition, $\underline{\lim}_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n \ge N\}$ and $\overline{\lim}_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n \ge N\}$. So for any fixed N, $\inf\{a_n : n \ge N\} \le \sup\{a_n : n \ge N\}$ by the definition of infimum and supremum of a set. Therefore, it is true for all N and $\underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n$.

Show that $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$ if and only if the sequence converges.
$\Leftarrow$ If the sequence converges, show that $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$.
There are three cases to consider: $\lim a_n = +\infty$, $\lim a_n = -\infty$, and $\lim a_n = a$.

Suppose $\lim a_n = +\infty$, it suffices to show that $\underline{\lim}_{n\to\infty} a_n = +\infty$. If $\lim a_n = +\infty$, then $\forall M > 0$, $\exists N$ s.t. $\forall n > N$, $a_n > M$. This implies that $\inf\{a_n : n \ge N\} > M$, and we have $\inf\{a_n : n \ge k\} \ge \inf\{a_n : n \ge N\} > M$, $\forall k > N$. Therefore, $\lim_{k\to\infty} \inf\{a_n : n \ge k\} = +\infty$, and $+\infty = \underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n = +\infty$.

Suppose $\lim a_n = -\infty$, it suffices to show that $\overline{\lim}_{n\to\infty} a_n = -\infty$. If $\lim a_n = -\infty$, then $\forall M < 0$, $\exists N$ s.t. $\forall n > N$, $a_n < M$. This implies that $\sup\{a_n : n \ge N\} < M$, and we have $\sup\{a_n : n \ge k\} \le \sup\{a_n : n \ge N\} < M$, $\forall k > N$. Therefore, $\lim_{k\to\infty} \sup\{a_n : n \ge k\} = -\infty$, and $-\infty = \underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n = -\infty$.

Suppose that $\lim a_n = a$, want to show that $\lim a_n \le \underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n \le \lim a_n$.
First show that $\overline{\lim}_{n\to\infty} a_n \le \lim a_n$. Since $a_n \to a$, $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n \ge N$, $|a_n - a| < \epsilon$. 2
This implies that $-\epsilon < a_n - a < \epsilon$ and $a_n < a + \epsilon$. Then $\sup\{a_n : n \ge N\} \le a + \epsilon$ and $\sup\{a_n : n \ge k\} \le a + \epsilon$, $\forall k \ge N$. Therefore, $\lim_{k\to\infty} \sup\{a_n : n \ge k\} \le a + \epsilon$, and $\overline{\lim}_{n\to\infty} a_n \le \lim a_n$.
Next show that $\lim a_n \le \underline{\lim}_{n\to\infty} a_n$. Since $a_n \to a$, $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n \ge N$, $|a_n - a| < \epsilon$. This implies that $-\epsilon < a_n - a < \epsilon$ and $a - \epsilon < a_n$. Then $a - \epsilon \le \inf\{a_n : n \ge N\}$ and $a - \epsilon \le \inf\{a_n : n \ge k\}$, $\forall k \ge N$. Therefore, $a - \epsilon \le \lim_{k\to\infty} \inf\{a_n : n \ge k\}$, and $\lim a_n \le \underline{\lim}_{n\to\infty} a_n$.
Therefore $\lim a_n \le \underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n \le \lim a_n$ and $\lim a_n = \underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$.

$\Rightarrow$ If $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n$, show that the sequence converges.
There are three cases to consider: $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = +\infty$, $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = -\infty$, and $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = a$.

Suppose $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = +\infty$, then $\forall M > 0$, $\exists N$ s.t. $\inf\{a_n : n \ge N\} > M$.
This implies that $\forall M > 0$, $\exists N$ s.t. $\forall n \ge N$, $a_n > M$ and $\lim a_n = +\infty$.
Suppose $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = -\infty$, then $\forall M < 0$, $\exists N$ s.t. $\sup\{a_n : n \ge N\} < M$.
This implies that $\forall M > 0$, $\exists N$ s.t. $\forall n \ge N$, $a_n < M$ and $\lim a_n = -\infty$.

Suppose $\underline{\lim}_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = a$. Since $\underline{\lim}_{n\to\infty} a_n = a$ then $\forall \epsilon > 0$, $\exists M_0$ s.t. $|a - \inf\{a_n : n > M_0\}| < \epsilon$ and $\inf\{a_n : n > M_0\} > a - \epsilon$. Therefore, $a - \epsilon < a_n$, $\forall n \ge M_0$. Since $\overline{\lim}_{n\to\infty} a_n = a$ then $\forall \epsilon > 0$, $\exists N_0$ s.t. $|a - \sup\{a_n : n > N_0\}| < \epsilon$ and $\sup\{a_n : n > N_0\} < a + \epsilon$. Therefore, $a_n < a + \epsilon$, $\forall n \ge N_0$.
Therefore $a - \epsilon < a_n < a + \epsilon$, $\forall n \ge \max\{N_0, M_0\}$ and $\lim a_n = a$.

We have shown $\underline{\lim}_{n\to\infty} a_n \le \overline{\lim}_{n\to\infty} a_n$, with equality holding if and only if the sequence converges.

III.19 Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of real numbers. Show that $\overline{\lim}_{n\to\infty}(a_n + b_n) \leq \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$, with equality holding if one of the original sequences converges.

For $a_n + b_n$, each are non-empty and bounded then $a_n \leq \sup a_n$ and $b_n \leq \sup b_n, \forall n$. Then $a_n + b_n \leq \sup a_n + \sup b_n, \forall n$. If it is true for all n, then it's true for the largest $\sup\{a_n + b_n\} \leq \sup a_n + \sup b_n$. If it's true for all n, then for any fixed N, $\sup\{a_n + b_n : n > N\} \leq \sup\{a_n : n > N\} + \sup\{b_n : n > N\}$ and surely $\lim_{N\to\infty} \sup\{a_n + b_n : n > N\} \leq \lim_{N\to\infty} \sup\{a_n : n > N\} + \lim_{N\to\infty} \sup\{b_n : n > N\}$. Therefore, $\overline{\lim}_{n\to\infty}(a_n + b_n) \leq \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$.

Assume without loss of generality that $a_n \to a$, then $\overline{\lim}_{n\to\infty} a_n = \underline{\lim}_{n\to\infty} a_n = \lim a_n$. From earlier homework we know that $\inf a_n + \sup b_n \leq \sup(a_n + b_n)$. Therefore, for any N $\inf\{a_n : n \geq N\} + \sup\{b_n : n \geq N\} \leq \sup\{a_n + b_n : n \geq N\}$, and $\underline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n \leq \overline{\lim}_{n\to\infty}(a_n + b_n)$. However, because $a_n \to a$, $\underline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n = \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$. Since $\underline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n \leq \overline{\lim}_{n\to\infty}(a_n + b_n) \leq \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$, $\overline{\lim}_{n\to\infty}(a_n + b_n) = \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$.

2

Problem 23: Prove that if V is a normed vector space and $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are convergent sequences of elements of V with limits a and b respectively, then $\lim(a_n + b_n) = a + b$ and $\lim(a_n - b_n) = a - b$ and if furthermore $c_1, c_2, c_3, \ldots$ is a sequence of real numbers converging to c, then $\lim c_n a_n = ca$

(a) Need to show that $\forall \epsilon > 0\ \exists N$ s.t. $\|(a_n + b_n) - (a + b)\| < \epsilon\ \forall n \geq N$. Since both $\{a_n\}$ and $\{b_n\}$ are both convergent sequences and the norm is just the distance between two points we have $\forall \epsilon > 0\ \exists N$ s.t. $\|a_n - a\| < \frac{\epsilon}{2}$ and $\|b_n - b\| < \frac{\epsilon}{2}$. Given any epsilon, choose $n \geq N$ and add the two inequalities together, $\|a_n - a\| + \|b_n - b\| < \epsilon$. Using property four of "normed vector spaces" page 63, we have $\|(a_n - a) + (b_n - b)\| \leq \|a_n - a\| + \|b_n - b\| < \epsilon \Rightarrow \|(a_n - a) + (b_n - b)\| < \epsilon$. Rearranging the terms on the lefthand side, $\|(a_n + b_n) - (a + b)\| < \epsilon\ \forall n \geq N$ as desired!

(b) Need to show $\lim(a_n - b_n) = a - b$. We can use the result from part (a) and re-write the above equations as $\lim(a_n + (-b_n)) = a + (-b) \Rightarrow \lim(a_n - b_n) = a - b$ as desired!

(c) Need to show $\forall \epsilon > 0\ \exists N$ s.t. $\|c_n a_n - ca\| < \epsilon\ \forall n \geq N$. First we know that convergent sequences are bounded and since both $\{a_n\}$ and $\{c_n\}$ are convergent sequences, $\|a_n\| < M$ and $\|c_n\| < M$. In addition since they both are convergent, given any epsilon positive $\exists N_1$ s.t. $\|a_n - a\| < \frac{\epsilon}{2M}$ and $\exists N_2$ s.t. $\|c_n - c\| < \frac{\epsilon}{2M}$. Let's choose $n \geq N = \max\{N_1, N_2\}$. Looking at the inequality $\|c_n a_n - ca\|$ we can re-write this as $\|c_n a_n - ac_n + ac_n - ca\|$ and rearranging the terms inside to get $\|c_n(a_n - a) + a(c_n - c)\| \Rightarrow$ (by the properties of normed spaces) $\leq |c_n| * \|a_n - a\| + |a| * \|c_n - c\|$ and since both $\{a_n\}$ and $\{c_n\}$ both converge and they are bounded by M we have $\|c_n a_n - ca\| \leq |c_n| * \|a_n - a\| + |a| * \|c_n - c\| < |M| * \frac{\epsilon}{2M} + |M| * \frac{\epsilon}{2M} = \epsilon\ \forall n \geq N$ as desired!

Additional Problems:

1. Show that $\lim s_n = +\infty$ if and only if $\lim(-s_n) = -\infty$.

Answer:

Since this problem contains an "if and only if" statement, we must show two logical directions. First let us show the "only if" direction. We are given that $\lim s_n = +\infty$ which means that $\forall M > 0, \exists N \ s.t. \ n \geq N \Rightarrow s_n > M$; we would like to show that $\forall M' < 0, \exists N \ s.t. \ n \geq N \Rightarrow -s_n < M'$. Given some $M' < 0$, set $M = -M'$. Since $M' < 0, M = -M' > 0$, so $\exists N \ s.t. \ n \geq N \Rightarrow s_n > M$. Multiplying the last inequality by -1 gives us that $-s_n < -M$ which in turn means that $-s_n < M'$ for every $n \geq N$. Thus for any $M' < 0$, there is some index after which all $-s_n$ are less than M'. Thus $\lim(-s_n) = -\infty$.

Now let us show the "if" direction. We are told that $\forall M' < 0, \exists N \ s.t. \ n \geq N \Rightarrow -s_n < M'$ and we would like to show that $\forall M > 0, \exists N \ s.t. \ n \geq N \Rightarrow s_n > M$. Given some $M > 0$, set $M' = -M$; since $M > 0$, we know that $M' < 0$, meaning that $\exists N$ such that $n \geq N \Rightarrow -s_n < M'$. Multiplying this inequality by -1 gives $s_n > -M'$; we know that $-M' = M$ so that $s_n > M$ for every $n \geq N$. Thus given any $M > 0$, there is some index after which all terms are greater than M so that $\lim s_n = +\infty$. Having shown both logical directions, the proof is complete.

2

2. Suppose that there exists a N_0 such that if $n \geq N_0$ then $s_n \leq t_n$.

(a) Prove that if $\lim s_n = +\infty$ then $\lim t_n = +\infty$.
If $\lim s_n = +\infty$, then $\forall M > 0$, $\exists N$ s.t. $\forall n \geq N$, $s_n > M$.
So given an $M > 0$ choose $N^* = \max\{N_0, N\}$, therefore $M < s_n \leq t_n$, $\forall n \geq N^*$. We have if $\lim s_n = +\infty$ then $\lim t_n = +\infty$.

(b) If $\lim s_n = s$ and $\lim t_n = t$, then $s \leq t$.
If $s_n \to s$, then $\forall \epsilon > 0$, $\exists N_s$ s.t. $\forall n \geq N_s$, $|s_n - s| < \epsilon$ and $s - \epsilon < s_n$. Likewise there exists a N_t.
Then given any $\epsilon > 0$, we choose $N^* = \max\{N_s, N_t, N_0\}$. Then we have if $s_n \leq t_n$, $s_n - t_n \leq 0$. From $s_n \to s$ and $t_n \to t$, $s - \epsilon - (t - \epsilon) < s_n - t_n \leq 0$, which implies that $s - t \leq 2\epsilon$. Since ϵ is arbitrary, $s \leq t$.

3. Show that $\liminf s_n = -\limsup(-s_n)$.

Proof:

2

We have shown that $\inf s_n = -\sup(-s_n)$ in Exam 1. Now $\liminf s_n = \lim_{N\to\infty} \inf\{s_n, n \geq N\} = \lim_{N\to\infty} -\sup\{-s_n, n \geq N\} = -\limsup_{N\to\infty} \sup\{-s_n, n \geq N\} = -\limsup(-s_n)$.

4. Let $\{s_n\}$ be a sequence of real numbers and let

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k.$$

(a) Show that $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$;

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim s_n = \lim \sigma_n$.

Proof:

(a) The second inequality is trivial. To prove the third inequality, we first

$$\sup_{n>M} \sigma_n \leq \frac{\quad}{M} + \sup_{k>N} s_k.$$

Since $n > M > N$, we can break σ_n into two parts:

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n} = \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n}.$$

Since $n > M$,

$$\frac{s_1 + \cdots + s_N}{n} \leq \frac{s_1 + \cdots + s_N}{M},$$

and

$$\frac{s_{N+1} + \cdots + s_n}{n} \leq \frac{(n-N)\sup_{k>N} s_k}{n} \leq \sup_{n>N} s_k.$$

2*

Therefore

$$\sigma_n \leq \frac{s_1 + \cdots + s_N}{M} + \sup_{k>N} s_k.$$

Immediately, we have

$$\sup_{n>M} \sigma_n \leq \frac{s_1 + \cdots + s_N}{M} + \sup_{k>N} s_k.$$

Now, fix N and let M goes to infinity, we have that

$$\limsup \sigma_n \leq \sup_{k>N} s_k$$

for all N. Finally, let $N \to \infty$, we have that

$$\limsup \sigma_n \leq \limsup s_k.$$

To prove the first inequality, use the fact that $\liminf s_n = -\limsup(-s_n)$. Since $\limsup(-\sigma_n) \leq \limsup(-s_n)$. $\Rightarrow$ $-\liminf \sigma_n \leq -\liminf s_n$ $\Rightarrow$ $\liminf s_n \leq \liminf \sigma_n$.

5. Prove that a sequence $\{s_n\}$ of real numbers is bounded if and only if $\limsup_{n\to\infty} |s_n| < +\infty$.

5.

Proof. $\{s_n\}$ bounded implies that $\sup\{|s_n| : n \in \mathbb{N}\}$ is bounded above by an upper bound, say M. Since $\sup\{|s_n| : n > N\} \leq \sup\{|s_n| : n \in \mathbb{N}\}$ for any N,

$$\limsup |s_n| = \lim_{N\to\infty} \sup\{|s_n| : n > N\} \leq \lim_{N\to\infty} \sup\{|s_n| : n \in \mathbb{N}\} \leq \sup\{|s_n| : n \in \mathbb{N}\} \leq M$$

If $\limsup |s_n| < +\infty$, there there is an $M > 0$ such that $\limsup |s_n| = M$, ie. for $\epsilon > 0$, there is an N' such that $|\sup\{|s_n| : n > N\} - M| < \epsilon$ for $N > N'$. Say $\epsilon = 1$. Then $\sup\{|s_n| : n > N\} < 1 + M$ for $N > N'$ so $\{|s_n| : n > N'\}$ is bounded. The set $\{|s_n| : n \leq N'\}$ is also bounded since it is a finite set. Therefore, $\{|s_n| : n \in \mathbb{N}\}$ is bounded so $\{s_n\}$ is a bounded sequence. 2*

□