1. *Discuss the continuity of the function* $f: R \to R$ *if*

(a) $f(x) = \begin{cases} 0 \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$

Proof: *We need to show that* $\forall \epsilon > 0, \exists \delta > 0$ *such that* $p \in E$(*or* R: *our metric space*)

and $d_x(p, p_0)\delta$, *then* $d_y(f(p), f(p_0)) < \epsilon$. *Let* $p_0 = 0$.

case 1) $p_0 < 0$: $d_x(p, 0) = p$ *and* $d_y(f(p), f(0)) = d_y(0,0) = 0$

case 2) $p_0 \leq 0$: $d_x(p, 0) = p$ *and* $d_y(f(p), f(0)) = d_y(p, 0) = p$.

V

Choose $\forall \epsilon > 0, \exists d_x(p, 0) = p < \delta$ *and* $d_y(p, 0) = p < \epsilon$.

(b) $f(x) = \begin{cases} x\sin\frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$

Proof : *We need to check if* $\lim_{x \to 0} f(x) = f(0)$ *so,* $\lim_{x \to 0} x\sin\frac{1}{x} = 0$.

case 1) *If x is nonzero,* $0 \leq \sin\frac{1}{x} \leq 1$, *and we have* $0 \leq \left|x\sin\frac{1}{x}\right| \leq |x|$ *for all nonzero x.*

by squeeze theorm, $\lim_{x \to 0} 0 \leq \lim_{x \to 0}\left|x\sin\frac{1}{x}\right| \leq \lim_{x \to 0}|x|$, *then* $0 \leq \lim_{x \to 0}\left|x\sin\frac{1}{x}\right| \leq 0$.

✓

Thus, $\lim_{x \to 0}\left|x\sin\frac{1}{x}\right| = 0$.

case 2) *If x is zero,* $\lim_{x \to 0} 0 = 0$.

Therefore, it is continuous.

Discussion:

$f(x) = x^2$ and $f(x) = 1$ are continuous functions.

Select $p = 0, q \neq 0$: $d(p = 0, q) = q$ and $d(f(p = 0), f(q)) = d(1, q^2) = |1 - q^2|$. The inequality $|1 - q^2| < \epsilon$ implies $-\epsilon < q^2 - 1 < \epsilon \implies \sqrt{1-\epsilon} < q < \sqrt{1+\epsilon}$.

Test with $\epsilon = \frac{1}{2}$: $\sqrt{1-\frac{1}{2}} < q < \sqrt{1+\frac{1}{2}} \implies \sqrt{\frac{1}{2}} < q < \sqrt{\frac{3}{2}}$. But we require a δ such that $q < \delta \implies \sqrt{\frac{1}{2}} < q < \sqrt{\frac{3}{2}}$. No selection of δ will satisfy both inequalities, so the function must be discontinuous at zero.

V ✗

(d) Claim: f is not continuous

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is not rational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where p and q are integers with no common divisors other than } \pm 1 \text{, and } q > 0 \end{cases}$$

Suppose that we want to check the continuity at some rational number $\frac{p}{q} = x_0$. Then $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. Let x be an irrational number such that $|x - x_0| < \delta$. This is possible by LUB 5 on page 26 in the text. Now given $\epsilon < \frac{1}{q}$, then we have $\forall \epsilon > 0\ |x - x_0| < \delta$ but $|f(x) - f(x_0)| = |0 - \frac{1}{q}| = |\frac{1}{q}| \geq \epsilon\ \forall \delta$ and irrational x.

✗

Rosenlicht 2

Let E, E' be metric spaces, $f : E \to E'$ a continuous function. Show that if S is a closed subset of E' then $f^{-1}(S)$ is a closed subset of E. Derive from this the results that if f is a continuous real-valued function on E then the sets $\{p \in E : f(p) \leq 0\}$, $\{p \in E : f(p) \geq 0\}$, $\{p \in E : f(p) = 0\}$ are closed.

2

Solution:

Need to Show: $\mathcal{C}(f^{-1}(S))$ is an open subset of E.

$S \subset E'$ closed $\implies$ $\mathcal{C}(S) \subset E'$ open. For continuous functions, the inverse image of an open subset is open, so $f^{-1}(\mathcal{C}(S)) \subset E$ is open. We desire to move the complement outside the inverse mapping. This is possible becuase $f^{-1}(\mathcal{C}(S)) = \{p \in E : f(p) \in \mathcal{C}(S)\} = \{p \in E : f(p) \notin S\} = \mathcal{C}(f^{-1}(S))$.

Now we have that $(f^{-1}(S))$ is closed. So the mapping $f : E \to \mathbb{R}$ maps closed subsets to closed subsets. Then $\{p \in E : f(p) \leq 0\}$, $\{p \in E : f(p) \geq 0\}$, $\{p \in E : f(p) = 0\}$ are closed subsets because they map to the closed subsets $\{x \in \mathbb{R} : x \leq 0\}, \{x \in \mathbb{R} : x \geq 0\}, \{0\}$.

9.a. Given some $x_1 \geq 0$ and some $\epsilon > 0$, we seek to find a $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies that $|\sqrt{x_1} - \sqrt{x_2}| < \epsilon$. Here, note that regardless of which is greater, we can say that

$$\sqrt{x_1} - \sqrt{x_2} = (\sqrt{x_1} - \sqrt{x_2}) \left(\frac{\sqrt{x_1} + \sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}}\right)$$ for $(x_1, x_2) \neq (0,0)$

First let us examine the case that $x_1 = 0$. Here, $|\sqrt{x_1} - \sqrt{x_2}| = \sqrt{x_2}$. Here, choosing $\delta < \epsilon^2$ gives us that $|x_1 - x_2| = x_2 < \epsilon^2$ so that $|\sqrt{x_1} - \sqrt{x_2}| = \sqrt{x_2} < \epsilon$. Thus we see that if $x_1 = 0$, the function $\sqrt{x}$ is continuous.

Now let us examine the case that $x_1 > 0$. Using the fact that

$$\sqrt{x_1} - \sqrt{x_2} = (\sqrt{x_1} - \sqrt{x_2}) \left(\frac{\sqrt{x_1} + \sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}}\right)$$

and distributing, we see that

$$\sqrt{x_1} - \sqrt{x_2} = \frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}}$$

Since $\sqrt{x_1} > 0$ and $\sqrt{x_2} > 0$, we know that $\sqrt{x_1} + \sqrt{x_2} > \sqrt{x_1}$ which lets us say that

$$\frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}} < \frac{x_1 - x_2}{\sqrt{x_1}}$$

Here, if we select $\delta = \sqrt{x_1} * \epsilon$, we see that $x_1 - x_2 < \sqrt{x_1} * \epsilon$ so that 2

$$\frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}} < \frac{x_1 - x_2}{\sqrt{x_1}} = \frac{\sqrt{x_1} * \epsilon}{\sqrt{x_1}} = \epsilon$$

Thus given any $x_1 \in \mathbb{R}$ and any $\epsilon > 0$, we see that $\exists \delta > 0$ such that $d(x_1, x_2) < \delta \Rightarrow d(\sqrt{x_1}, \sqrt{x_2}) < \epsilon$ and thus that $\sqrt{x}$ is continuous $\forall x \in \mathbb{R}$.

9.b. Here we note that

$$x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$$

which lets us say that

$$\frac{x - 1}{\sqrt{x} - 1} = \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{\sqrt{x} - 1} = \sqrt{x} + 1$$

From this we see that

$$\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1} = \lim_{x \to 1} \sqrt{x} + 1 = 2$$

13. *Write down the details of the following alternate proof that a continuous real valued function f on a compact metrix space E is bounded and attains a maximum :*

If f is not bounded, then for $n = 1,2,3 \ldots$ there is a point $p_n \in E$ such that $|f(p_n)| > n$,

and a contradiction arises from the existence of a convergent subsequence of $p_1, p_2, p_3, \ldots$

Thus f is bounded and we can find a sequence of points $q_1, q_2, q_3, \ldots$ of E

such that $\lim\limits_{n\to\infty} f(p_n) = l.u.b\,\{f(p): p \in E\}$.

A maximum will be attained by f at the limit of a convergent subsequence of $q_1, q_2, q_3, \ldots$

Proof: Assume that f is not bounded, then for $n = 1,2,3 \ldots$

there is a point $p_n \in E$ such that $|f(p_n)| > n$.

Since E is compact, $\exists$ a convergent subsequence $\{p_{n_k}\}$ with $\lim\limits_{k\to\infty} p_{n_k} = p_0$, $p_0 \in E$.

Now because f is continuous, we can use the property of continuity. $\lim\limits_{k\to\infty} f(p_{n_k}) = f(p_0)$.

Then we could find a positive integer N such that $\forall k > N$,

$1 > |f(p_{n_k}) - f(p_0)| \geq |f(p_{n_k})| - |f(p_0)| > n_k - |f(p_0)|$ 2

$\Rightarrow 1 + |f(p_0)| > n_k$ but we cannot. which is contradiction.

Therefore f is bounded and nonempty. It imples that we can find a sequence in $f(E)$ where

$\mathrm{Lim}_{n\to\infty} f(q_n) = \mathrm{l.u.b.}\{f(p): p \in E\}$.

Since E is compact, $\exists$ a convergent subsequence $\{q_{n_k}\}$ of $\{q_n\}$ where $\lim\limits_{n\to\infty} q_{n_k} = q_0$, $q_0 \in E$.

However, since $\lim\limits_{n\to\infty} f(q_n) = \lim\limits_{k\to\infty} f(q_{n_k})$, we must have $f(q_0) = l.u.b\,\{f(p): p \in E\}$.

Thus, $f(q_0) \geq f(p), \forall p \in E$, $f(q_0)$ is max.

14. (a) *Prove that if S is a nonempty compact subset of a metrix space E and* $p_0 \in E$ *then* $\min\{d(p_0, p): p \in S\}$ *exists (distance from* p_0 *to S).*

Proof: Let $f(x) = d(p, p_0) = |p - p_0|$ where $f: E \to R$.

This is continuous function since if we choose $\delta = \epsilon$,

$|f(p) - f(p_1)| = |d(p, p_0) - d(p_1, p_0)| \le d(p, p_1) < \epsilon$ so $d(p, p_1) < \epsilon$ then $|f(p) - f(p_1)| < \epsilon$.

also, if $f: E \to E'$ is continuous and S is a subset of E,

then the restriction of f to S is continuous on S, too. Therefore, $f(x)$ is continuous on S.

Since f is a continuous real valued function on a nonempty compact space S,

f attains a minimum at some point by Corollary 2 on textbook p78.

(b) *Prove that if S is a nonempty closed subset of* E^n *and* $p_0 \in E^n$ *then* $\min\{d(p_0, p): p \in S\}$ *exists.*

2

Proof:

If $p_0 \in E$ then $\min\{d(p_0, p): p \in S\} = 0$, we can assume that $p_0 \in S^c$. Choose $\epsilon > 0$,

such that the closed ball $\overline{B(p_0, \epsilon)} \subset E^n$ contains points in S, or $\overline{B(p_0, \epsilon)} \cap S = \emptyset$.

Consider the continuous function $f: E^n \to \mathbb{R}$ given by the function $f(p) = d(p_0, p)$.

Then f is continuous on $\overline{B(p_0, \epsilon)} \cap S$ as well.

Observe that $\overline{B(p_0, \epsilon)} \cap S$ is closed and bounded subset of E^n. and it is compact.

by Corollary 2 on textbook p78, f attains a munimum at some point in $\overline{B(p_0, \epsilon)} \cap S$

or $\min\{d(p_0 . p): p \in S\}$

15. Prove that for any nonempty compact metric space E, $\max\{d(p,q) : p,q \in E\}$ exists.

Proof:

Since E is compact and $\{d(p,q) : p,q \in E\}$ is bounded and nonempty, E must be bounded. Then we can find a sequence of points $\{(p_n, q_n)\}_{n=1,2,...}$ of E s.t.

$$\lim_{n\to\infty} d(p_n, q_n) = \sup\{d(p,q) : p,q \in E\}.$$

Since E is comapct, $\exists$ convergent subsequences $\{p_n\}, \{q_n\}$ where $\{p_{n_k}\}, \{q_{n_k}\}$ converge to some point $p_o, q_0 \in E$. Thus, we have that

$$d(p_0, q_0) = \lim_{k\to\infty} d(p_{n_k}, q_{n_k})$$

$$= \lim_{n\to\infty} d(p_n, q_n) = \sup\{d(p,q) : p,q \in E\}.$$

Hence, $\max\{d(p,q) : p,q \in E\}$ exists. $\square$

2

10. Discuss the continuity of the function $f : E^2 \to \mathbb{R}$ if

(a) $f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(b) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(c) $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Proof:

(a) By Lemma and Proposition on pg.75-76 in section 4.3, we know that $f(x,y)$ is continuous when $(x,y) \neq (0,0)$. NTS (rather check) continuity at $(0,0)$. Then by the last Proposition in section 4.2 on pg. 74, which was also used on prob.2, if $f(x,y)$ is continuous at $(0,0)$, then $\forall\{p_n\}_{n\geq 1}$ converges to $(0,0)$ we will have $\lim_{n\to\infty} f(p_n) = f(0,0) = 0$. Thus, choose $p_n = (\frac{1}{n}, \frac{1}{n})$, so then we have that $\lim_{n\to\infty} p_n = (0,0)$ and

$$f(p_n) = \frac{1}{(\frac{1}{n})^2 + (\frac{1}{n})^2} = 2n^2.$$

2

So we see that as $n \to \infty$, we have $f(p_n) \to \infty$, which is a contradiction. Hence $f(x,y)$ is not continuous at $(0,0)$. $\square$

(b) Using the same idea as (a), NTS continuity at $(0,0)$. Choose $\{p_n\}_{n\geq 1}$ converges to $(0,0)$ as $p_n = (\frac{1}{n}, \frac{1}{n})$. Then

$$\lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} \frac{\frac{1}{n}\frac{1}{n}}{(\frac{1}{n})^2 + (\frac{1}{n})^2} = \frac{1}{2} \neq 0 = f(0,0).$$

A contradiction, hence $f(x,y)$ is not continous at $(0,0)$. $\square$

(c) NTS continuity at $(0,0)$. By definition of continuity, $\forall\varepsilon > 0$, if $\exists\delta > 0$, s.t. $\forall p = (x_p, y_p)$ $d(p,(0,0)) = \sqrt{x_p^2 + y_p^2} < \delta$ where $d(f(p),0) = |f(p)| < \varepsilon$.

$f(x,y)$ is continuous at $(0,0)$. Check.

$$|f(p)| = |\frac{x_p y_p^2}{x_p^2 + y_p^2}| \leq |\frac{x_p y_p^2}{2x_p y_p}| = \frac{|y_p|}{2} < \varepsilon \iff |y_p| < 2\varepsilon.$$

Since $x_p^2 \geq 0$, let $\delta = 2\varepsilon$, then for $\sqrt{x_p^2 + y_p^2} < \delta$, $|y_p| < 2\varepsilon$, $|f(p)| < \varepsilon$. Hence $f(x,y)$ is continuous at $(0,0) \implies f(x,y)$ is continuous on E^2. $\square$