17. Is the function x^2 uniformly continuous on $\mathbb{R}$? The function $\sqrt{|x|}$? Why?

Let us show that x^2 is not uniformly continuous by showing that $\forall\epsilon > 0\ \forall\delta > 0,\ \exists x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|x^2 - y^2| \geq \epsilon$. To do this, choose $x = \frac{\epsilon}{\delta}$ and set $y = \frac{\epsilon}{\delta} + \frac{\delta}{2}$. We see that $|x - y| = |\frac{\epsilon}{\delta} - \frac{\epsilon}{\delta} - \frac{\delta}{2}| = \frac{\delta}{2}$. However we also see that $|x^2 - y^2| = \left|\left(\frac{\epsilon}{\delta} + \frac{\delta}{2}\right) - \left(\frac{\epsilon}{\delta}\right)^2\right| = \left|\frac{\epsilon^2}{\delta^2} + \epsilon + \frac{\delta}{4} - \frac{\epsilon^2}{\delta^2}\right| = \left|\epsilon + \frac{\delta^2}{4}\right| = \epsilon + \frac{\delta^2}{4} > \epsilon$. Thus $\forall\epsilon > 0\ \forall\delta > 0\ \exists x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|x^2 - y^2| \geq \epsilon$ and the function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Since $f(x) = \sqrt{|x|}$ is the composition of two functions (namely $\sqrt{x}$ and $|x|$) we can show that each of these functions is uniformly continuous and use the result from question **20** to conclude that f is also uniformly continuous. First, let us show that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$. To do this, we must find a $\delta > 0$ which is a function of ϵ only such that $|x - y| < \delta$ implies that $|\sqrt{x} - \sqrt{y}| < \epsilon$. Let us choose $\delta = \epsilon^2$. Then $|x - y| < \delta$ implies that $|x - y| < \epsilon^2$. Now there are two possibilities here: either that $x - y < 0$ or $x - y > 0$. Say $x - y < 0$ so that $|x - y| = y - x$. Then we have $y - x < \epsilon^2$ and thus that $y < x + \epsilon^2$. Since $y > 0$ and $x + \epsilon^2 > 0$, we can take the square root of both sides while still preserving the inequality, giving us $\sqrt{y} < \sqrt{x + \epsilon^2}$. By definition, for any $u, v > 0$ we know that $\sqrt{u + v} \leq \sqrt{u} + \sqrt{v}$. Here, this lets us say that $\sqrt{y} < \sqrt{x + \epsilon^2}$ implies that $\sqrt{y} < \sqrt{x} + \epsilon$. Subtracting $\sqrt{x}$ from both sides then gives $\sqrt{y} - \sqrt{x} < \epsilon$ whenever $|x - y| < \delta = \epsilon^2$.

Now let us consider the case that $x - y > 0$. Then $|x - y| = x - y > 0$ and we have $0 < x - y < \delta = \epsilon^2$. Adding y gives $y \leq x < y + \epsilon^2$. Taking the square root gives $\sqrt{y} \leq \sqrt{x} < \sqrt{y + \epsilon^2}$. Again using the aforementioned property of inequalities containing square roots, we may now say that $\sqrt{y} \leq \sqrt{x} < \sqrt{y} + \epsilon$. Subtracting $\sqrt{y}$ then gives $0 \leq \sqrt{x} - \sqrt{y} < \epsilon$. Combining this inequality with the one from end the of the preceding paragraph gives us that $|\sqrt{x} - \sqrt{y}| < \epsilon$ whenever $|x - y| < \delta$ for every $x, y \in [0, \infty)$. Thus $\sqrt{x}$ is uniformly continuous.

For the function $|x|$ we can show uniform continuity by selecting $\delta = \epsilon$. Here, $|x - y| < \delta$ implies $|x - y| < \epsilon$ and, using the reverse triangle inequality we see that $|f(x) - f(y)| = ||x| - |y|| \leq |x - y| < \epsilon$ thus giving us $||x| - |y|| < \epsilon$. Thus $\sqrt{x}$ and $|x|$ are both uniformly continuous and their composition is as well.

18. Prove that for any metric space E, the identity function on E is uniformly continuous.

The identity function is defined as $Id(x) = x$, $\forall x \in E$. Here we need to show that given any $\epsilon > 0$, $\exists\delta > 0$ such that $\forall x, y \in E, d(x, y) < \delta$ implies that $d(Id(x), Id(y)) < \epsilon$. Given some $\epsilon > 0$, let us set $\delta = \epsilon$. Then, taking any $x, y \in E$ such that $d(x, y) < \delta$, using $\delta = \epsilon$ we can say that $d(x, y) < \epsilon$. Now, since $Id(x) = x$ and $Id(y) = y$, we may substitute these into the preceding inequality, giving us $d(Id(x), Id(y)) < \epsilon$. Thus we have shown that $d(x, y) < \delta$ implies that $d(Id(x), Id(y)) < \epsilon$ and the proof is complete.

19. Prove that for any metric space E and any $p_0 \in E$, the real-valued function sending any p into $d(p_0, p)$ is uniformly continuous.

Let $f(p) = d(p_0, p)$ where p_0 is any point in E. Here we wish to show that f is unformly continuous on E. To do this, we must show that given any $\epsilon > 0$, $\exists \delta > 0$ such that $\forall q, r \in E$, $d(q,r) < \delta$ implies that $d(f(q), f(r)) < \epsilon$. Here, note that, via the triangle inequality, $d(q,r) \geq |d(q,p_0) - d(p_0,r)|$. This means that $\delta > d(q,r)$ implies that $\delta > |d(q,p_0) - d(p_0,r)|$. Since $f(q) = d(q,p_0)$ and $f(r) = d(r,p_0)$, we see that the preceding statement implies that $\delta > |f(q) - f(r)|$. Thus here, given any $\epsilon > 0$, setting $\delta = \epsilon$ gives us that $d(q,r) < \delta$ implies $d(f(q), f(r)) < \epsilon$ for every $q, r \in E$ and the proof for uniform continuity is complete.

20. State precisely and prove: A uniformly continuous function of a uniformly continuous function is uniformly continuous.

Here let $f : E \to E'$ and $g : E' \to S$. We seek to show that $h = g \circ f : E \to S$ is uniformly continuous if f and g are. To do this, we must show that given any $\epsilon_h > 0$ $\exists \delta_h > 0$ such that $d_E(p,q) < \delta_h$ implies that $d_S(h(p), h(q)) < \epsilon_h$, $\forall p, q \in E$.

Now, since g is uniformly continuous, given some $\epsilon_h > 0$ we know that $\exists \delta_g$ such that $\forall r, s \in E'$, $d_{E'}(r,s) < \delta_g$ implies that $d_S(g(r), g(s)) < \epsilon_h$. Using the fact that f is uniformly continuous, we can set $\epsilon_f = \delta_g$ and we know that 1 $\forall u, v \in E$ we have $d_E(u,v) < \delta_f$ implies that $d_{E'}(f(u), f(v)) < \epsilon_f = \delta_g$ which in turn implies that $d_S(g(f(u)), g(f(v))) < \epsilon_h$. Rewriting this last inequality, we have $d_S((g \circ f)(u), (g \circ f)(v)) < \epsilon_h$ whenever $d_E(u,v) < \delta_f$. Thus given any $\epsilon_h > 0$ we can set $\delta_h = \delta_f$ and be guaranteed that $d_E(u,v) < \delta_f$ implies that $d_S(h(u), h(v)) < \epsilon_h$ $\forall u, v \in E$, meaning that $h : E \to S$ is uniformly continuous and the proof is complete.

(1) $\exists \delta_f$ such that

Problem 22

Assume the norm on V is $\|\cdot\|_1$ and the one on V' is $\|\cdot\|_2$.

a.) Assume f is continuous at a point x_0.

Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$\forall z \in V, \quad \|z - x_0\|_1 < \delta \Rightarrow \|f(z) - f(x_0)\|_2 < \varepsilon$

Assume $\|x - y\|_1 < \delta$ for some $x, y \in V$.

One has $\|x-y\|_1 = \|x - y + x_0 - x_0\|_1$.

Thus $\|x - y + x_0 - x_0\|_1 < \delta$ and

$\|f(x-y+x_0) - f(x_0)\|_2 < \varepsilon$.

That is $\|f(x) - f(y) + f(x_0) - f(x_0)\|_2 < \varepsilon$ and so

$\|f(x) - f(y)\|_2 < \varepsilon$.

Hence $\|x-y\|_1 < \delta \Rightarrow \|f(x) - f(y)\|_2 < \varepsilon$.

So f is uniformly continuous and then continuous everywhere.

6.1 $\Rightarrow$ Assume $A = \{ \frac{\|f(x)\|}{\|x\|}, x \in V, x \neq 0 \}$

is bounded above say by $M > 0$.

For $x \neq 0$, $\|f(x)\| = \|x\| \frac{\|f(x)\|}{\|x\|} \leq M\|x\|$.

for $x = 0$ we also have $\|f(x)\| = 0 \leq M\|x\| = 0$

Hence $\forall x \in V$, $\|f(x)\| \leq M\|x\|$.

Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{M}$.

$\|x\| < \delta \Rightarrow M\|x\| < \varepsilon \Rightarrow \|f(x)\| < \varepsilon$.

Hence f is continuous at zero and therefore is continuous everywhere.

$\Leftarrow$ Assume f is continuous. Let us show that A is bounded.

Since f is continuous at 0, $\forall \varepsilon > 0, \exists \delta > 0$ such that $\|x\| < \delta \Rightarrow \|f(x)\| < \varepsilon$.

For $x \neq 0$, $x = \frac{2\|x\|}{\delta} \cdot \frac{\delta}{2\|x\|} x$

since $\left\| \frac{\delta}{2\|x\|} x \right\| = \frac{\delta}{2} < \delta$, $\left\| f\left(\frac{\delta}{2\|x\|} x \right) \right\| < \varepsilon$.

Thus $\frac{\|f(x)\|}{\|x\|} = \frac{\frac{2\|x\|}{\delta}}{\|x\|} \left\| f\left(\frac{\delta}{2\|x\|} x \right) \right\| < \frac{2}{\delta} \varepsilon$

So A is bounded below by 0 and above by $\frac{2\varepsilon}{\delta}$.

c) Let $(v_1, \dots, v_n)$ be a basis of V.

Consider the function $\varphi : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto \|\sum x_i v_i\|$

φ is a norm on $\mathbb{R}^n$. Indeed:

$\varphi(X) = 0 \Leftrightarrow \varphi(X_1, \dots, X_n) = 0 \Leftrightarrow \|\sum x_i v_i\| = 0$

$\Leftrightarrow \sum x_i v_i = 0$

$\Leftrightarrow x_i = 0 \quad i = 1, \dots, n$

$\Leftrightarrow X = 0.$

$$\begin{aligned}\varphi(X+Y) &= \|\sum (x_i + y_i) v_i\| \\ &= \|\sum x_i v_i + \sum y_i v_i\| \\ &\leq \|\sum x_i v_i\| + \|\sum y_i v_i\| \\ &= \varphi(X) + \varphi(Y).\end{aligned}$$

$$\begin{aligned}\varphi(\lambda X) = \|\sum \lambda x_i v_i\| &= |\lambda| \, \|\sum x_i v_i\| \\ &= |\lambda| \, \varphi(X).\end{aligned}$$

Since φ is a norm it is continuous on $\mathbb{R}^n$.

Let $\|\cdot\|_0$ denote the Euclidean norm.

$S = \{x \in \mathbb{R}^n, \ \|x\|_0 = 1\}$ is compact.

Thus $\exists x_0 \in S$ such that $m = \varphi(x_0) \leq \varphi(x)$.

Remark that $m \neq 0$ since $m = 0 \Rightarrow \varphi(x_0) = 0$ and $x_0 = 0$ impossible as $\|x_0\|_0 = 1$

For $x \in \mathbb{R}^n$, $x \neq 0$,

$\frac{x}{\|x\|_0} \in S'$; $m \leq \varphi\left(\frac{x}{\|x\|_0}\right)$

$$m \leq \left\| \sum \frac{x_i}{\|x\|_0} v_i \right\|_1$$

$$\|x\|_0 \, m \leq \left\| \sum_{i=1}^{n} x_i v_i \right\|_1 = \|x\|_1$$

Now for $x = \sum_{i=1}^{n} x_i v_i \neq 0$

$$\begin{aligned} \| f(\textstyle\sum \alpha_i v_i) \| &= \left\| \sum_{i=1}^{n} x_i f(v_i) \right\| \\ &\leq \sum_{i=1}^{n} |x_i| \, \| f(v_i) \| \\ &\leq \sum_{i=1}^{n} \|x\|_0 \, \| f(v_i) \| \\ &\leq \sum_{i=1}^{n} \frac{1}{m} \|x\|_1 \, \| f(v_i) \| \end{aligned}$$

Thus $\frac{\|f(x)\|}{\|x\|} \leq \sum_{i=1}^{n} \frac{1}{m} f(v_i)$

Thus f is continuous by part b.).

Checking that the norm makes the set of infinite sequences above a normed vector space:

1. Given that at least one of the x_i's is nonzero, clearly $\max\{|x_1|, |x_2|, |x_3|, \dots\} > 0$.

2. $\max\{|x_1|, |x_2|, |x_3|, \dots\} = 0$ only if all the x_i's are zero.

3. $\|(cx_1, cx_2, cx_3, \dots)\| = \max\{|cx_1|, |cx_2|, |cx_3|, \dots\} = \max\{|c||x_1|, |c||x_2|, |c||x_3|, \dots\} = |c| \cdot \max\{|x_1|, |x_2|, |x_3|, \dots\}$.

4. By the triangle inequality, $|x_i + y_i| \leq |x_i| + |y_i| \leq \max_i(x_i) + \max_i(y_i)$. So we must have $\max_i(x_i + y_i) \leq \max(x_i) + \max(y_i)$.

Showing the map is a one-to-one linear transformation:

Within each component, we have $f(x_i) = i \cdot x_i$. This is clearly a linear transformation, and one-to-one. Since all components are then mapped by a linear transformation, the larger map is a linear transformation.

Assume $(x_1, 2x_2, 3x_3, \dots) = (x_1', 2x_2', 3x_3', \dots)$

Then $i x_i' = i x_i$ for $i = 1, 2, \dots$

so $x_i = x_i'$

and $(x_1, x_2, x_3, \dots) = (x_1', x_2', \dots)$

So the map is one-to-one.

Set $X^{n} = (x_1^n, x_2^n, \ldots)$

where $x_i^n = \begin{cases} 0 & \text{if } i \neq n \\ 1 & \text{if } i = n \end{cases}$

ie $x^1 = (1, 0, 0, 0 \ldots)$

$x^2 = (0, 1, 0, 0, \ldots)$

$x^3 = (0, 0, 1, 0, \ldots)$.

If f denotes the map: $(x_1, x_2, \ldots) \mapsto (x_1, 2x_2, 3x_3, \ldots)$.

$f(x^n) = (x_1^n, 2x_2^n, \ldots,) = (0, \ldots, 0, n, 0, 0 \ldots)$

Thus $\|f(x^n)\| = n$ and $\|x^n\| = 1$.

So $\dfrac{\|f(x^n)\|}{\|x^n\|} = n$

and the set $\left\{ \dfrac{\|f(x)\|}{\|x\|}, x \neq 0 \right\}$ is not bounded

and then f is not

23. Use Problem 22 to prove that if V is a finite dimensional vector space over $\mathbb{R}$ and $|| \, ||_1, || \, ||_2$ are two norm funtions on V.

Solution

Let e_n be the canomial base of V, and let $x \in V$. Then $x = \sum_{i=1}^{n} x_i e_i =$ $||x||_1 = ||\sum_{i=1}^{n} x_i e_i||_1 \le \sum_{i=1}^{n} |x_i| * ||e_i||_1$ by using the definition of a norm and $|| \, ||_1$ is any norm. We can apply the Cauchy Schwarz inequality. $||x||_1 \le \sum_{i=1}^{n} |x_i| * ||e_i||_1 \le \sqrt{\sum_{i=1}^{n} |x_i|^2} * \sqrt{\sum_{i=1}^{n} ||e_i||_1^2}$. We can see that $\sqrt{\sum_{i=1}^{n} |x_i|^2} = ||x||_e$ and $\sqrt{\sum_{i=1}^{n} ||e_i||_1^2} = \mu_1$ where $|| \, ||_e$ is the euclidean norm, and μ_1 is some constant. Then, $||x||_1 \le \mu_1 * ||x||_e$. Now we show that $f(x) = ||x||_1$ is continuous with respect to the euclidean norm. Let $\epsilon > 0$, we need to show

$\exists \delta_\epsilon > 0$ s.t. $||x - y||_e < \delta_\epsilon = || \, ||x||_1 - ||y||_1 \, || < \epsilon$. Let $\delta_e = \frac{\epsilon}{|\mu_1|}$. Then $\frac{\epsilon}{|\mu_1|} \ge ||x-y||_e \ge || \, ||x||_e - ||y||_e \, || = \frac{\epsilon}{|\mu_1|} \ge ||\frac{||x||_1}{\mu_1} - \frac{||y||_1}{\mu_1}||_e = \frac{1}{\mu_1} * || \, ||x||_1 - ||y||_1 \, ||$. Then, $\epsilon > || \, ||x||_1 - ||y||_1 \, ||_\epsilon =$ F is continuous. Let $S = \{x \in v \text{ s.t. } ||x||_e = 1\}$. S is closed and bounded, so it is compact , and then F has a minimum value in S at x_{min}. Now $\forall x \in v$, $\frac{x}{||x||_e} \in s = ||\frac{x}{||x||_e}||_1 \ge m_1$ where m_1 is the minimum value of F in s. Then, $||x||_1 \ge m_1 * ||x||_e$. For any norms, we have basically shown that $\exists \mu_1, m_2, \mu_2$ such that $m_1 * ||x||_e \le ||x||_1 \le \mu_1 * ||x||_e = m_e * ||x||_e \le ||x||_2 \le \mu_2 ||x||_e = \exists m = \frac{m_1}{m_2}$ and $\mu = \frac{\mu_1}{\mu_2}$ such that $m \le \frac{||x||_1}{||x||_2} < \mu$. It basically follows that given that $\mathbb{R}$ is complete, and all norms are equivalent to the euclidean norm, the space is complete $\forall$ norm.

33)

a. Show that the sequence of functions $x, x^2, x^3, \ldots$ converges uniformly on $[0, a]$ for any $a \in (0,1)$, but not on $[0,1]$.

Let $\{f_n\} = \{x^n\}$, and suppose $f^n \to f$. We must show that for $\epsilon > 0$, $\exists N$ such $d(f, f^n) < \epsilon$ whenever $n > N$ for all x.

For $a \in (0,1)$, it is clear to see that $x^n \to 0$ as n approaches infinity. We must then show $|x^n| < \epsilon$ whenever n is greater than some N.

On $[0, a]$, x^n attains its max at $x = a$, so $x^n < a^n$. Then note a^n decreases with increasing n, so we choose N such $a^N < \epsilon$.

$\{f_n\}$ doesn't converge uniformly on $[0,1]$ because at $x = 1$ $f^n = (1)^n = 1 \neq 0$ for all n.

b. Show that the sequence of functions $x(1-x), x^2(1-x), x^3(1-x), \ldots$ converges uniformly on $[0,1]$.

Since on $[0,1]$, at least one of the quantities x^n and $(1-x)$ is less than 1, and at is at most 1, thus we might guess $f = 0$. Then we must show for $\epsilon > 0$, $\exists N$ such $|x^n(1-x) - 0| < \epsilon$ if $n > N$.

x^n and $(1-x)$ are both continuous functions, and using calculus, we can calculate the maximum value that $|x^n(1-x)|$ attains on $[0,1]$.

$$\frac{d}{dx}\left(x^n(1-x)\right) = -x^n + nx^{n-1}(1-x) = -x^n + nx^{n-1} - nx^n = -x + n - nx = 0$$

So $x^n(1-x)$ attains its max at $x = \frac{n}{n+1}$, which is $\left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right)$

Then $|x^n(1-x)| < \left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right) < \frac{1}{n+1} < \epsilon$. If we choose $N = (1-\epsilon)/\epsilon$, then whenever $n > N$, we will have $|x^n(1-x)| < \epsilon$. ✓

34) Is the sequence of functions $f_1, f_2, f_3, \ldots$ on $[0,1]$ uniformly convergent if

$f_n(x) = \frac{x}{1+nx^2}$? Note $f_n \to 0$. We must find for $\epsilon > 0$, an N such $n > N$ implies $|f_n - 0| < \epsilon$.

Observe $f_n' = \frac{(1+nx^2)-x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2} = 0$. This has solution at $x = \sqrt{\frac{1}{n}}$, and f_n attains a max value $\frac{1}{2}\sqrt{\frac{1}{n}}$. So $\left|\frac{x}{1+nx^2}\right| < \frac{1}{2}\sqrt{\frac{1}{n}}$, thus if we choose $N > 1/4\epsilon^2$, we will have $|f_n - 0| < \epsilon$ whenever $n > N$.

$f_n(x) = \frac{nx}{1+nx^2}$? Note for all n, $f_n(0) = 0$, and for $x > 0$, $f_n \to \frac{1}{x}$. Since $\lim f_n$ is not continuous on $[0,1]$, it does not converge uniformly.

$f_n(x) = \frac{nx}{1+n^2x^2}$? Again we have $f_n \to 0$. We must find for $\epsilon > 0$, an N such $n > N$ implies $|f_n - 0| < \epsilon$.

Again, taking the derivative and setting it to 0 gives us:

$$f_n' = \frac{(1+n^2x^2)n - nx(2n^2x)}{(1+n^2x^2)^2} = \frac{n - n^3x^2}{(1+n^2x^2)^2} = 0$$

This has solution at $x = 1/n$. But this means that f_n attains a max of $f_n\left(\frac{1}{n}\right) = \frac{n\left(\frac{1}{n}\right)}{1+n^2\left(\frac{1}{n}\right)^2} = \frac{1}{2}$. Thus it would not be possible to choose an N for all ϵ, specifically any $\epsilon < \frac{1}{2}$.

37) Let $f_1, f_2, f_3, \ldots$ and $g_1, g_2, g_3, \ldots$ be uniformly convergent sequences of real-valued functions on a metric space E. Show that the sequence $f_1 + g_1, f_2 + g_2, \ldots$ is uniformly convergent.

Let $f_n \to f$ and $g_n \to g$. We have that for all x, for $\epsilon > 0$, $\exists N_1, N_2$ such $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ whenever $n > N_1$ and $|g_n(x) - g(x)| < \frac{\epsilon}{2}$ whenever $n > N_2$.

We hypothesize that $f_n + g_n \to f + g$. So we must find N such $|(f_n + g_n)(x) - (f+g)(x)| < \epsilon$ whenever $n > N$. Note $|(f_n + g_n)(x) - (f+g)(x)| = |(f_n - f)(x) + (g_n - g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$. So if we take $N = \max(N_1, N_2)$, we will have that $|(f_n +$

(37) Con't'd

$g_n)(x) - (f+g)(x)| \le |f_n(x) - f(x)| + |g_n(x) - g(x)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $\{f_n + g_n\}$ is uniformly convergent.

How about $f_1g_1, f_2g_2, \ldots$? Suppose that $f_ng_n \to fg$. We must then find N such $|(f_ng_n)(x) - (fg)(x)| < \epsilon$ whenever $n > N$, for all $\epsilon > 0$, and x.

$$\begin{aligned}|(f_ng_n)(x) - (fg)(x)| &= |(f_ng_n)(x) - (fg_n)(x) + (fg_n)(x) - (fg)(x)| \\ &\le |(f_n - f)(x)||g_n(x)| + |f(x)||(g_n - g)(x)|\end{aligned}$$

We can make $|(f_n - f)(x)|$ and $|(g_n - g)(x)|$ arbitrarily small, but the behavior $g_n(x)$ and $f(x)$ are undetermined. So no, $\{f_ng_n\}$ is not guaranteed to be uniformly convergent. But we can make it so with the addition that all f_n, g_n, as well as f, g are bounded. Then $|g_n(x)|$ and $|f(x)|$ can be replaced with constants, and we can choose N accordingly.

counterexample: $f_n = g_n = x + \frac{1}{n}$; $f = g = x$

(38) Given $\epsilon > 0$, there exists some N such that $d(f_n(x), f(x)) < \epsilon$ when $n > N$. Since each function is bounded, all elements of $f_n(x)$ are bounded and thus all elements exists in an open ball $B_r(x_0)$. Which means that we have $d(f(x), y) \le d(f(x), f_n(x)) + d(f_n(x), x_0) < \epsilon + r$, for Thus all elements are contained in an open ball $B_{\epsilon+r}(x_0)$. Therefore the sequence is bounded.