1.

(a)

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\frac{1}{x}$ is a differentiable function on $\mathbb{R}\backslash\{0\}$, and $\sin(x)$ is a differentiable function on $\mathbb{R}$. Hence, $\sin(\frac{1}{x})$ is a differentiable function on $\mathbb{R}\backslash\{0\}$ since it is a composite function of two differentiable functions. x is a differentiable function on $\mathbb{R}\backslash\{0\}$, so $x\sin(\frac{1}{x})$ is a differentiable function on $\mathbb{R}\backslash\{0\}$. So we're interested in seeing if $f(x)$ is differentiable at $x = 0$.

Applying the definition of derivatives directly, we see if

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

has a limit.

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h\sin(\frac{1}{h})}{h} = \lim_{h\to 0} \sin(\frac{1}{h})$$

This limit does not exist. More specifically, this expression is equivalent to

$$\lim_{t\to\infty} \sin t$$

the value of which fluctuates between $[-1, 1]$ but never converges. So $f(x)$ is not differentiable at $x = 0$, and is differentiable only on $\mathbb{R}\backslash\{0\}$.

(b)

$$f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is differentiable on $\mathbb{R}\backslash\{0\}$ similar to the function in (a). Again, we check to see if this function is at $x = 0$.

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2\sin(\frac{1}{h})}{h} = \lim_{h\to 0} h\sin(\frac{1}{h})$$

For any given real number z, $|\sin(z)| \leq 1$. Hence,

$$\left|h\sin(\frac{1}{h})\right| \leq |h|$$

Then we know that

$$\lim_{h\to 0} \left|h\sin(\frac{1}{h})\right| \leq \lim_{h\to 0} |h| = 0,$$

therefore we have

$$\lim_{h\to 0} h\sin(\frac{1}{h}) = 0.$$

Hence the limit exists, meaning $f'(0)$ exists and the f is differentiable everywhere.

(c)

$$f(x) = \sqrt{|x|}$$

$g(x) = |x|$, $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}^+ \cup \{0\}$ is a differentiable function , and $h(x) = \sqrt{x}$ is a differentiable function on $\mathbb{R}^+$. Hence, $f(x) = h(g(x))$ is a differentiable function on $\mathbb{R}\backslash\{0\}$. We're interested in the differentiability of the function at $x = 0$.

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{\sqrt{|h|}}{h} = \lim_{h\to 0}\pm\sqrt{\frac{|h|}{h^2}} = \lim_{h\to 0}\pm\sqrt{\frac{|h|}{|h|^2}} = \lim_{h\to 0}\pm\sqrt{\frac{1}{|h|}} = \pm\infty$$

The $\pm$ sign is there to remind ourselves that the limit can approached from both right and left. The limit does not exist, so this function is not differentiable at $x = 0$. $f(x)$ is differentiable on $\mathbb{R}\backslash\{0\}$.

2.

(a)

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}\\
&= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{2h}\\
&= \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{2h} - \frac{f(x_0-h)-f(x_0)}{2h}\right)\\
&= \frac{1}{2}\lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h} + \frac{f(x_0-h)-f(x_0)}{-h}\right)\\
&= \frac{1}{2}\left(\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} + \lim_{h\to 0}\frac{f(x_0-h)-f(x_0)}{-h}\right)\\
&= \frac{1}{2}\left(f'(x_0) + \lim_{t\to 0}\frac{f(x_0+t)-f(x_0)}{t}\right)\\
&= \frac{1}{2}\big(f'(x_0)+f'(x_0)\big) = f'(x_0)
\end{aligned}$$

Where $t = -h$. The assumption that $f(x)$ is differentiable at x_0 was used.

(b)
$\alpha, \beta \in \mathbb{R}$, compute

$$\lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0+\beta h)}{h}$$

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0+\beta h)}{h}\\
&= \lim_{h\to 0}\frac{f(x_0+\alpha h)-f(x_0)+f(x_0)-f(x_0+\beta h)}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x_0+\alpha h)-f(x_0)}{h} - \frac{f(x_0+\beta h)-f(x_0)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{f(x_0+\alpha h)-f(x_0)}{h} - \frac{f(x_0+\beta h)-f(x_0)}{h}\right)\\
&= \lim_{h\to 0}\left(\alpha\frac{f(x_0+\alpha h)-f(x_0)}{\alpha h} - \beta\frac{f(x_0+\beta h)-f(x_0)}{\beta h}\right)\\
&= \lim_{h\to 0}\alpha\frac{f(x_0+\alpha h)-f(x_0)}{\alpha h} - \lim_{h\to 0}\beta\frac{f(x_0+\beta h)-f(x_0)}{\beta h}\\
&= \alpha f'(x_0) - \beta f'(x_0)\\
&= (\alpha-\beta)f'(x_0)
\end{aligned}$$

3) Proof of the chain rule:

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x))-g(f(x_0))}{x-x_0} = \lim_{x \to x_0} \left(\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)} \cdot \frac{f(x)-f(x_0)}{x-x_0} \right)$$

$$= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0)$$

a. What's wrong with this proof?

The problem is having $f(x) - f(x_0)$ as a factor in the denominator. This quantity might be zero for values of x arbitrarily close to x_0. Example: $f(x) = x \sin\left(\frac{1}{x}\right)$, as in problem 1a.

b. Alter this slightly into a correct proof.

We introduce $A(x, x_0) = \begin{cases} \frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)}, & x \neq x_0 \\ g'(f(x_0)), & x = x_0 \end{cases}$. This way, we can have $\lim_{x \to x_0} A(x, x_0) = g'(f(x_0))$.

Making the appropriate substitutions into the faulty proof, it becomes

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x))-g(f(x_0))}{x-x_0} = \lim_{x \to x_0} \left(\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)} \cdot \frac{f(x)-f(x_0)}{x-x_0} \right)$$

$$= \lim_{x \to x_0} A(x, x_0) \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0))f'(x_0)$$

4.

Proof. Let f be a differentiable on (a, b) and $f'(x) \geq 0$ for $x \in (a, b)$. Take $x, y \in (a, b)$ such that $y > x$. Then by the mean value theorem, there exists a $c \in (x, y)$ such that $f(y) - f(x) = (y - x) \cdot f'(c)$. $f'(c) \geq 0 \implies f(y) \geq f(x)$ and so f is increasing. Similarly, if $f'(x) \leq 0$ for every $x \in (a, b)$, we get that f is decreasing by reversing the above inequalities.

Now suppose f is differentiable and increasing on (a, b). Then $y \geq x \implies f(y) \geq f(x)$ so

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} \geq 0 \implies f'(x) \geq 0$$

for all $x \in (a, b)$. Similarly, if f is strictly decreasing we get

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} \leq 0 \implies f'(x) \leq 0$$

for all $x \in (a, b)$.

6. Prove that a differentiable real-valued function on $\mathbb{R}$ with bounded derivative is uniformly continuous.

Proof: Let f be a differentiable real-valued function on $\mathbb{R}$ with a bounded derivative. By the Mean Value Theorem, for any $a, b \in \mathbb{R}$ (wlog let $a < b$) there exist a $c \in (a, b)$ where $\frac{f(b)-f(a)}{(b-a)} = f'(c)$. Also because f has a bounded derivative, there exists some constant $N > 0$ where $|f'(x)| \leq N$ for all $x \in \mathbb{R}$. By combining these two facts, we get that $\left|\frac{f(b)-f(a)}{(b-a)}\right| \leq N$ or $|f(b) - f(a)| \leq N|b - a|$. Pick any $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{N}$. Then whenever $|b - a| < \varepsilon$, $|f(b) - f(a)| < N\delta = \varepsilon$. Therefore f is uniformly continuous.

7 Let $a, b \in \mathbb{R}, a < b$, and let f be a differentiable real-valued function on an open subset of $\mathbb{R}$ that contains $[a, b]$. Show that if γ is any real number between $f'(a)$ and $f'(b)$ then there exists a number $c \in (a, b)$ such that $\gamma = f'(c)$

Define the real valued function g by the following rule $g(x) = f(x) - \gamma x$. Since the sum of differentiable functions is also differentiable we know that $g(x)$ is also differentiable and is defined as $g'(x) = f'(x) - \gamma$. Moreover, we add that $g(x)$ is continuous because the sum of continuous functions is continuous. Since g is continuous over a compact set then we know that it attains a maximum and a minimum at some point in the interval $[a, b]$. However, the minimum does not occur at a or b since by the construction of g it is evident that $g'(a) < 0$ and $g'(b) > 0$. Hence g attains its minimum at some point $c \in (a, b)$ such that $g'(c) = 0$. This means $g'(c) = 0 = f'(c) - \gamma$ and therefore $f'(c) = \gamma$ and we are done.

8. Let $a, b \in \mathbb{R}, a < b$, and let f, g be continuous real-valued functions on $[a, b]$ that are differentiable on (a, b). Prove that there exists a number $c \in (a, b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

(*Hint:* Consider the function

$$F(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a)).)$$

Proof

We consider the function $F(x)$ as stated above. Since both f and g are differentiable on (a, b), then by the proposition on page 101 of IA the derivative of $F(x)$ is given by

$$F'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)).$$

Since $F(a) = F(b) = 0$, continuous and real-valued on $[a, b]$ and differentiable on (a, b) by Rolle's theorem (page 104 of IA) there exists a $c \in \mathbb{R}$ so that $F'(c) = 0$. Hence we find

$$F'(c) = 0 = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a))$$

or

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)). \quad \square$$

9. (a)

Let a be an extremity of U. To be more specific, let $a = g.l.b.U$. The proof for the case $a = l.u.b.U$ will be similar.

Since f and g are differentiable on U, where U is an open interval $(a, b), x < b$ for all x of interest. f, g are continuous. Cauchy mean value theorem states that for $a < x$, $x \in U$, if f, g are continuous real-valued functions on $[a+h, x]$ and differentiable on $(a+h, x)$, there exists $c \in (a+h, x)$ such that

$$f'(c)\big(g(x) - g(a+h)\big) = g'(c)\big(f(x) - f(a+h)\big)$$

We are given that neither g nor g' are zero on U. Therefore,

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a+h)}{g(x) - g(a+h)}$$

The right-hand side is well-defined since the left hand side is well-defined (g, g' never zero. In the case $g(x) - g(a+h) = 0$ in the limit $h \to 0$, we must have $f(x) - f(a+h) = 0$ in the limit, which is given). Taking limit on both sides,

$$\lim_{h \to 0} \frac{f'(c)}{g'(c)} = \lim_{h \to 0} \frac{f(x) - f(a+h)}{g(x) - g(a+h)}$$

$$\lim_{h \to 0} \frac{f'(c)}{g'(c)} = \lim_{h \to 0} \frac{f(x) - f(a+h)}{g(x) - g(a+h)} = \frac{\lim_{h \to 0}(f(x) - f(a+h))}{\lim_{h \to 0}(g(x) - g(a+h))} = \frac{\lim_{h \to 0} f(x)}{\lim_{h \to 0} g(x)} = \lim_{h \to 0} \frac{f(x)}{g(x)}$$

The above used the fact that the limit operation can be distributed because the numerator and the denominator converge individually, and that $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} g(a+h) = 0$. Since $h \to 0$, in the limit, $c \in (a, x)$. The previous expression is equivalent to

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

for some $c \in (a, x)$. Taking the limit again on both sides,

$$\lim_{x \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

We know that $a < c < x$. Since $|a - x| \to 0$ as $x \to a$, we can write

$$\lim_{x \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Which is the expression we were looking for.

14) Use Taylor's theorem to prove the "binomial theorem" for positive integral exponent n:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}x^3 + \cdots + x^n$$

Proof: According to Taylor's theorem, if f is $n+1$ times differentiable on an open interval U, then for any $\alpha, \beta \in U$ we have

$$f(\beta) = f(\alpha) + \frac{f'(\alpha)}{1!}(\beta-\alpha) + \frac{f''(\alpha)}{2!}(\beta-\alpha)^2 + \cdots + \frac{f^{(n)}(\alpha)}{n!}(\beta-\alpha)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(\beta-\alpha)^{n+1}$$

Consider $f(x) = (a+x)^n$. Clearly, f is $n+1$ times differentiable, with $f^{(n+1)}(c) = 0$ for all c. Since the a, b are arbitrary, take $\beta = x$ and $\alpha = 0$ in the formation of Taylor's theorem to get

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$\Rightarrow (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}x^3 + \cdots + x^n$$

As desired.