MATH4318

Homework 2

1) Suppose that $a, b \in \mathbb{R}$ with $a < b$. Fix $N \in \mathbb{N}$ and let $P_N = \{x_k\}_{k=0}^{N}$ denote the partition of $[a, b]$ with $x_k = a + k\frac{(b-a)}{N}$. Compute the upper and lower Darboux sums associated to the function $f(x)$ and the partition P_N when:

(a) $f(x) = x$

We first note that f is a strictly increasing function and is bounded on $[a, b]$, thus supremums occur at right endpoints, and infimums at left endpoints.

$$U(f, P_N) = \sum_{k=1}^{N} \sup(f, [x_{k-1}, x_k]) \cdot (x_k - x_{k-1}) = \sum_{k=1}^{N} f(x_k) \cdot (x_k - x_{k-1}) = \sum_{k=1}^{N} x_k \cdot (x_k - x_{k-1})$$
$$= \sum_{k=1}^{N} \left(a + k\frac{b-a}{N}\right) \cdot \left(\frac{b-a}{N}\right) = \sum_{k=1}^{N} \frac{a(b-a)}{N} + \sum_{k=1}^{N} k\frac{(b-a)^2}{N^2}$$
$$= \frac{Na(b-a)}{N} + \frac{(b-a)^2}{N^2}\frac{N(N+1)}{2} = a(b-a) + \frac{(b-a)^2}{2}\frac{N+1}{N}$$

$$L(f, P_N) = \sum_{k=1}^{N} \inf(f, [x_{k-1}, x_k]) \cdot (x_k - x_{k-1}) = \sum_{k=1}^{N} f(x_{k-1}) \cdot (x_k - x_{k-1}) = \sum_{k=1}^{N} x_{k-1} \cdot (x_k - x_{k-1})$$
$$= \sum_{k=1}^{N} \left(a + (k-1)\frac{b-a}{N}\right) \cdot \left(\frac{b-a}{N}\right) = \sum_{k=1}^{N} \frac{a(b-a)}{N} + \sum_{k=1}^{N} (k-1)\frac{(b-a)^2}{N^2}$$
$$= \frac{Na(b-a)}{N} + \frac{(b-a)^2}{N^2}\frac{(N-1)N}{2} = a(b-a) + \frac{(b-a)^2}{2}\frac{N-1}{N}$$

(b) $f(x) = x^3$

We first note that f is a strictly increasing function and is bounded on $[a, b]$, thus supremums occur at right endpoints, and infimums at left endpoints.

$$U(f, P_N) = \sum_{k=1}^{N} \sup(f, [x_{k-1}, x_k]) \cdot (x_k - x_{k-1}) = \sum_{k=1}^{N} f(x_k) \cdot (x_k - x_{k-1}) = \sum_{k=1}^{N} x_k^3 \cdot (x_k - x_{k-1})$$
$$= \sum_{k=1}^{N} \left(a + k\frac{b-a}{N}\right)^3 \cdot \left(\frac{b-a}{N}\right)$$
$$= \sum_{k=1}^{N} \left(a^3 + 3ka^2\frac{b-a}{N} + 3k^2a\left(\frac{b-a}{N}\right)^2 + \left(k\frac{b-a}{N}\right)^3\right) \cdot \left(\frac{b-a}{N}\right)$$
$$= \sum_{k=1}^{N} \left(a^3\left(\frac{b-a}{N}\right) + 3ka^2\left(\frac{b-a}{N}\right)^2 + 3k^2a\left(\frac{b-a}{N}\right)^3 + k^3\left(\frac{b-a}{N}\right)^4\right)$$
$$= a^3(b-a) + \frac{3a^2(b-a)^2}{N}\frac{(N+1)}{2} + \frac{3a(b-a)^3}{N^2}\frac{(N+1)(2N+1)}{6} + \frac{(b-a)^4}{N^2}\frac{(N+1)^2}{4}$$

$$L(f,P_N)=\sum_{k=1}^{N}\inf(f,[x_{k-1},x_k])\cdot(x_k-x_{k-1})=\sum_{k=1}^{N}f(x_{k-1})\cdot(x_k-x_{k-1})=\sum_{k=1}^{N}x_{k-1}^3\cdot(x_k-x_{k-1})$$

$$=\sum_{k=1}^{N}\left(a+(k-1)\frac{b-a}{N}\right)^3\cdot\left(\frac{b-a}{N}\right)$$

$$=\sum_{k=1}^{N}\left(a^3+3(k-1)a^2\frac{b-a}{N}+3(k-1)^2a\left(\frac{b-a}{N}\right)^2+\left((k-1)\frac{b-a}{N}\right)^3\right)\cdot\left(\frac{b-a}{N}\right)$$

$$=\sum_{k=1}^{N}\left(a^3\left(\frac{b-a}{N}\right)+3(k-1)a^2\left(\frac{b-a}{N}\right)^2+3(k-1)^2a\left(\frac{b-a}{N}\right)^3+(k-1)^3\left(\frac{b-a}{N}\right)^4\right)$$

$$=a^3(b-a)+\frac{3a^2(b-a)^2}{N}\frac{(N-1)}{2}+\frac{3a(b-a)^3}{N^2}\frac{(N-1)(2N-1)}{6}+\frac{(b-a)^4}{N^2}\frac{(N-1)^2}{4}$$

2.

Proof. Suppose that U_n is the upper sum corresponding to partition P_n and suppose that L_n is the lower sum corresponding to partition Q_n. Take any $\epsilon > 0$ and choose N large enough such that $U_N - L_N = U(f,P_N) - L(f,Q_N) < \epsilon$ which is possible since the limit of $U_n - L_n = 0$. Then for the partition $P = P_N \cup Q_N$ we have $U(f,P) \leq U(f,P_N)$ and $L(f,Q_N) \leq L(f,P)$. Thus we have $U(f,P) - L(f,P) \leq U(f,P_N) - L(f,Q_N) < \epsilon$. Thus, f satisfies the condition of Darboux integrability and is therefore integrable which implies $U(f) = L(f)$.

Now take any $\epsilon > 0$ and choose N such that $U_n - L_n < \epsilon$ for $n \geq N$. We know that $U(f) = L(f)$ which is the supremum of all of the lower Darboux sums, so $U_n - U(f) \leq U_n - L_n < \epsilon$ for $n \geq N$ Thus, (U_n) converges to $U(f)$. Similarly, we know that $L(f) + U(f)$ which is the infimum of all upper Darboux sums, so $L(f) - L_n \leq U_n - L_n < \epsilon$ for $n \geq N$. Therefore L_n converges to $L(f)$. Thus,

$$\int_a^b f(x)dx = \lim_n U_n = \lim_n L_n$$

□

Problem 3

Proof:
Since the cardinality of $Q \backslash P = 1$, viewed as a set this implies there is one element in Q that is not in P. Suppose that this extra element lies between the $i-1$'th and i'th element and let us denote it as $x^\star \in (x_{i-1}, x_i)$. We need to show that $U(f;Q) \leq U(f;P)$ or $U(f;P) - U(f,Q) > 0$. If we expand the summations and write out this difference we get

$$\begin{aligned} U(f;P) - U(f,Q) &= \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) \\ &\quad - \sup_{x\in[x_{i-1},x^\star]} f(x)(x^\star - x_{i-1}) - \sup_{x\in[x^\star,x_i]} f(x)(x_i - x^\star) \\ &= \left(\sup_{x\in[x_{i-1},x_i]} f(x)(x^\star - x_{i-1}) + \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x^\star) \right) \\ &\quad - \left(\sup_{x\in[x_{i-1},x^\star]} f(x)(x^\star - x_{i-1}) + \sup_{x\in[x^\star,x_i]} f(x)(x_i - x^\star) \right) \\ &= \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \sup_{x\in[x_{i-1},x^\star]} f(x) \right) (x^\star - x_{i-1}) \\ &\quad + \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \sup_{x\in[x^\star,x_i]} f(x) \right) (x_i - x^\star). \end{aligned}$$

Since $\sup_{x\in A} f(x) \leq \sup_{x\in B} f(x)$ for $A \subset B$, we have

$$\begin{aligned} U(f;P) - U(f,Q) &= \overbrace{\left(\sup_{x\in[x_{i-1},x_i]} f(x) - \sup_{x\in[x_{i-1},x^\star]} f(x) \right)}^{>0} \overbrace{(x^\star - x_{i-1})}^{>0} \\ &\quad + \underbrace{\left(\sup_{x\in[x_{i-1},x_i]} f(x) - \sup_{x\in[x^\star,x_i]} f(x) \right)}_{>0} \underbrace{(x_i - x^\star)}_{>0} > 0. \end{aligned}$$

We can now show that the inequality still holds for the case in which the cardinality of $Q \backslash P = k > 1$ (with $P \subset Q$) by using induction, i.e. since its true at the base ($k = 1$), we assume true for $k-1$ and test at k. We need to show that $U(f;P) - U(f;Q) > 0$ when P and Q differ by k elements. We can define k nested partitions which differ from each other by exactly one element, i.e. $P = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_{k-1} \subset P_k = Q$. Then

$$\begin{aligned} U(f;P) - U(f;Q) &= U(f;P_0) - U(f;P_k) \\ &= U(f;P_0) + (U(f;P_{k-1}) - U(f;P_{k-1})) - U(f;P_k) \\ &= \underbrace{(U(f;P_0) - U(f;P_{k-1}))}_{\substack{>0 \\ \text{by induction hypothesis}}} + \underbrace{(U(f;P_{k-1}) - U(f;P_k))}_{\substack{>0 \\ \text{since cardinality } P_k \backslash P_{k-1} = 1}} > 0. \end{aligned}$$

4) Show that if f is integrable on $[a, b]$ and if $[c, d] \subset [a, b]$ then f is integrable on $[c, d]$.

Proof: Since f is integrable there exists a partition $P = \{x_0 = a, x_1, x_2, \dots, x_N = b\}$ of $[a, b]$ such that for every $\epsilon > 0$, $U(f, P) - L(f, P) < \epsilon$.

Consider $Q = P \cup \{c, d\}$. Since $P \subset Q$, $U(f, Q) \leq U(f, P)$, and $L(f, Q) \geq L(f, P)$, or $-L(f, Q) \leq -L(f, P)$. Adding, we get $U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P) < \epsilon$.

Rewrite $U(f, Q) = U(f, A) + U(f, B) + U(f, C)$ and $L(f, Q) = L(f, A) + L(f, B) + L(f, C)$, where $A = \{x \in P \mid a \leq x_i \leq c\}$, $B = \{x \in P \mid c \leq x_i \leq d\}$, and $C = \{x \in P \mid d \leq x_i \leq b\}$. Then $U(f, Q) - L(f, Q) = [U(f, A) - L(f, A)] + [U(f, B) - L(f, B)] + [U(f, C) - L(f, C)] \leq U(f, P) - L(f, P) < \epsilon$. Since each bracketed term is nonnegative, we have individually each is less than ϵ. This shows there exists a partition of $[c, d]$ such the difference between upper and lower Darboux sum is less than any ϵ, specifically, B.

5.

We need to show that given any $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ with $mesh(P) < \delta$, we have $U(g, P) - L(g, P) < \epsilon$.

Let $\epsilon > 0$ be given. Since f is integrable on $[a, b]$, there exists $\delta_1 > 0$ such that for any partition P of $[a, b]$ with $mesh(P) < \delta_1$,

$$U(f, P) - L(f, P) < \frac{\epsilon}{2}$$

Choose δ_1 so small that there is at most one point y in each interval in the partition where $f(y) \neq g(y)$. To be more specific, define

$$P = \{a = x_0 < x_1 < \cdots < x_N = b\}$$

with $mesh(P) < \delta_1$, where δ_1 is chosen to be so small that there exists at most one point $y_i \in [x_{i-1}, x_i]$ where $f(y_i) \neq g(y_i)$ for $i = 1, 2, \dots, N$. This is possible since the number of such points is finite. Since there are finitely many points y at which $f(y) \neq g(y)$, let T be the total number of these points, where we know $T \leq N$ due to our choice of δ_1. Let Q be the set of all the integers k such that the interval $[x_{k-1}, x_k]$ has a point y such that $f(y) \neq g(y)$. It is given that g is bounded, and f must be bounded since it is integrable. Let

$$M_f = \sup_{x \in [a,b]} f(x), \quad m_f = \inf_{x \in [a,b]} f(x)$$

$$M_g = \sup_{x \in [a,b]} g(x), \quad m_g = \inf_{x \in [a,b]} g(x).$$

We know $M_g - m_f \geq 0$ and $M_f - m_g \geq 0$ since f and g differ at finitely many points. Let us compute an upper bound for $U(g, P)$.

$$\begin{aligned}
U(g,P) &= U(f,P) - \sum_{i\in Q} \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) + \sum_{i\in Q} \sup_{x\in[x_{i-1},x_i]} g(x)(x_i - x_{i-1}) \\
&\leq U(f,P) - \sum_{i\in Q} m_f(x_i - x_{i-1}) + \sum_{i\in Q} M_g(x_i - x_{i-1}) \\
&= U(f,P) + \sum_{i\in Q}(M_g - m_f)(x_i - x_{i-1}) < U(f,P) + \sum_{i\in Q}(M_g - m_f)\delta \\
&= U(f,P) + T(M_g - m_f)\delta
\end{aligned}$$

Where δ is a positive number to be a new upper bound for the mesh of partition P. Similarly,

$$\begin{aligned}
L(g,P) &= L(f,P) - \sum_{i\in Q} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) + \sum_{i\in Q} \inf_{x\in[x_{i-1},x_i]} g(x)(x_i - x_{i-1}) \\
&\geq L(f,P) - \sum_{i\in Q} M_f(x_i - x_{i-1}) + \sum_{i\in Q} m_g(x_i - x_{i-1}) \\
&= L(f,P) - \sum_{i\in Q}(M_f - m_g)(x_i - x_{i-1}) > L(f,P) - \sum_{i\in Q}(M_f - m_g)\delta \\
&= L(f,P) - T(M_f - m_g)\delta
\end{aligned}$$

Therefore,

$$\begin{aligned}
U(g,P) - L(g,P) &< U(f,P) - L(f,P) + T(M_g - m_f)\delta + T(M_f - m_g)\delta \\
&= U(f,P) - L(f,P) + T(M_f + M_g - m_f - m_g)\delta
\end{aligned}$$

Choose

$$\delta = \min(\delta_1, \frac{\epsilon}{2T(M_f + M_g - m_f - m_g)})$$

and have $mesh(P) < \delta$. Then

$$\begin{aligned}
U(g,P) - L(g,P) &< U(f,P) - L(f,P) + T(M_f + M_g - m_f - m_g)\delta \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

6. Let f be a bounded function on $[a,b]$ with $|f(x)| \leq B$ for all $x \in [a,b]$.

   a. Show that for all partitions P of $[a,b]$ that

$$U(f^2: P) - L(f^2: P) \leq 2B(U(f: P) - L(f: P).$$

Proof: Let $P = \{a, x_1, \dots, x_{n-1}, b\}$ be a partition of $[a,b]$ and consider

$$U(f^2;P) - L(f^2;P) = \sum_{k=1}^{n} M(f^2,[x_{k-1},x_k])(x_k - x_{k-1}) - \sum_{k=1}^{n} m(f^2,[x_{k-1},x_k])\,(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n}(x_k - x_{k-1})[M(f^2,[x_{k-1},x_k]) - m(f^2,[x_{k-1},x_k])]$$

$$= \sum_{k=1}^{n}(x_k - x_{k-1})[M(f,[x_{k-1},x_k])^2 - m(f,[x_{k-1},x_k])^2]$$

$$= \sum_{k=1}^{n}(x_k - x_{k-1})[M(f,[x_{k-1},x_k]) - m(f,[x_{k-1},x_k])]\,[M(f,[x_{k-1},x_k]) + m(f,[x_{k-1},x_k])]$$

$$\le \sum_{k=1}^{n}(x_k - x_{k-1})[M(f,[x_{k-1},x_k]) - m(f,[x_{k-1},x_k])]\ 2B\ (*)$$

$$= \sum_{k=1}^{n} 2B\,[M(f,[x_{k-1},x_k])(x_k - x_{k-1}) - m(f,[x_{k-1},x_k])(x_k - x_{k-1})]$$

$$= 2B\big(U(f;P) - L(f;P)\big).$$

$(*)$ Because $|f(x)| \le B$ both $M(f,[x_{k-1},x_k])$ and $m(f,[x_{k-1},x_k])$ are at most B.

b. Show that if f is integrable on $[a,b]$ then f^2 is also integrable on $[a,b]$.

Proof: Let $\varepsilon > 0$ be given. Because f is integrable $\exists\delta$ such that when $mesh(P) < \delta$, $U(f;P) - L(f;P) < \frac{\varepsilon}{2B}$. Now consider

$$U(f^2;P) - L(f^2;P) \le 2B[U(f;P) - L(f;P)],$$

from the first part of this problem. Continuing

$$U(f^2;P) - L(f^2;P) \le 2B[U(f;P) - L(f;P)] < 2B\frac{\varepsilon}{2B} = \varepsilon.$$

Therefore, f^2 is integrable.

c. Let f and g be integrable functions on $[a,b]$. Show that fg is integrable on $[a,b]$.

Hint: Express fg as a linear combination of $(f+g)^2$ and $(f-g)^2$.

Proof: Notice: $fg = \frac{1}{4}(f-g)^2 - \frac{1}{4}(f+g)^2$. Based on the part b of this question along with lemmas proved in class, f and g being integrable implies that $f+g$ and $f-g$ are integrable. Then $f-g$ and $f+g$ integrable implies $(f-g)^2$ and $(f+g)$^2 are integrable. Next $(f-g)^2$ and $(f+g)^2$ integrable implies that $\frac{1}{4}(f-g)^2 - \frac{1}{4}(f+g)^2$ is integrable. Since $fg = \frac{1}{4}(f-g)^2 - \frac{1}{4}(f+g)^2$, then fg is integrable!

7.

For bounded function f on $[a, b]$, we wish to prove that

$$|U(f) - L(f)| \leq |U(f, P) - L(f, P)|$$

for all partitions P of $[a, b]$.

For any partition P of $[a, b]$, we know that $U(f, P) \geq L(f, P)$. Therefore, $U(f, P) - L(f, P) \geq 0$ which means $|U(f, P) - L(f, P)| = U(f, P) - L(f, P)$. The inequality to be proven in the problem statement is then equivalent to

$$|U(f) - L(f)| \leq U(f, P) - L(f, P)$$

We note that

$$U(f) = \inf_P U(f, P), \quad L(f) = \sup_P L(f, P) \tag{0.0.1}$$

Also, we proved in class that for any partition P and Q of $[a, b]$, $U(f, P) \geq L(f, Q)$. Since this holds for any partitions P and Q, we have $U(f) \geq L(f)$. so this means $|U(f) - L(f)| = U(f) - L(f)$. The problem statement is then equivalent to

$$U(f) - L(f) \leq U(f, P) - L(f, P).$$

From $U(f) = \inf_P U(f, P)$, we deduce that $U(f) \leq U(f, P)$. Similarly, from $L(f) = \sup_P L(f, P)$, we deduce that $L(f) \geq L(f, P)$. Changing signs around, we have $-L(f) \leq -L(f, P)$. Adding inequalities side by side, we have

$$U(f) - L(f) \leq U(f, P) - L(f, P)$$

which is what we needed to show.